

Infinity computations in cellular automaton forest-fire model

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Abstract

Recently a number of traditional models related to the percolation theory has been considered by means of a new computational methodology that does not use Cantor's ideas and describes infinite and infinitesimal numbers in accordance with the principle 'The whole is greater than the part' (Euclid's Common Notion 5). Here we apply the new arithmetic to a cellular automaton forest-fire model which is connected with the percolation methodology and in some sense combines the dynamic and the static percolation problems and under certain conditions exhibits critical fluctuations. It is well known that there exist two versions of the model: real forest-fire model where fire catches adjacent trees in the forest in the step by step manner and simplified version with instantaneous combustion. Using new approach we observe that in both situations we deal with the same model but with different time resolution. We show that depending on the "microscope" we use the same cellular automaton forest-fire model reveals either instantaneous forest combustion or step by step firing. By means of the new approach it was also observed that as far as we choose an infinitesimal tree growing rate and infinitesimal ratio between the ignition probability and the growth probability we determine the measure or extent of the system size infinity that provides the criticality of the system dynamics. Correspondent inequalities for grosspowers are derived.

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1 Introduction

Recently a new applied point of view on infinite and infinitesimal numbers has been introduced in [22, 27, 31]. The new approach does not use Cantor's ideas (see [7]) and describes infinite and infinitesimal numbers that are in accordance with the principle 'The whole is greater than the part' (Euclid's Common Notion 5). It gives a possibility to work with finite, infinite, and infinitesimal quantities *numerically* by using a new kind of computers – the Infinity Computer – introduced in [23, 24, 32, 33]. In our previous paper [2] we applied the new computational tools to study percolation phase transition. It has been established that in an infinite system phase transition point is not really a point as with respect of traditional approach. We showed that in light of new arithmetic it appears as a critical interval, rather than a critical point. Depending on the "microscope" we use this interval could be regarded as either finite or infinite or infinitesimal interval. Using new approach we observed that in vicinity of percolation threshold we have many different infinite clusters instead of one infinite cluster that appears in traditional consideration. Moreover, we have now a tool to distinguish those infinite clusters. In particular, we can distinguish spanning infinite clusters from embedded infinite clusters.

In this paper, we are going to apply the new arithmetic to a cellular automaton forest-fire model [12, 13] which is tightly connected with the percolation methodology and in some sense combines the dynamic and the static percolation problems. Forest-fire model elegantly represents the simplest example of a model that under certain conditions exhibits critical fluctuations. Actually, community that deals with the forest-fire model and its applications, distinguishes between two versions of the model: real forest-fire model where fire catches adjacent trees in the forest in the step by step manner and a simplified version with the instantaneous combustion [12, 14]. Using the new approach we show that in both situations we deal with the same model but with different time resolution. We observe that depending on "microscope" we use the same cellular automaton forest-fire model reveals either instantaneous forest combustion or step by step firing.

Another interesting feature of the forest-fire model is that the model is critical when driven in a certain limit. That is, critical behavior will occur in the limit of slowly growing trees. Though slow-growing, the trees must grow quickly compared to the time interval between spontaneously ignited fires, i.e., the ratio between the ignition probability and the growth probability should be moved toward zero. It should be emphasized that the limits discussed imply infinite model system size, so-called thermodynamic limit. Finite model size limit results in some difficulties during cellular automaton forest-fire model implementation in numerical experiment. The new computational approach proposed recently in [22, 27] allows us to overcome this difficulties. As far as we choose infinitesimal trees growing rate and infinitesimal ratio between the ignition probability and the growth probability we determine the measure or extent of the system size infinity that provides the criticality of the system dynamics. By means of the new approach we derive correspondent inequalities for grosspowers that control trees growing rate infinitesimality, infinitesimal ignition probability and the system size infinity.

The outline of the paper is as follows. In Sec. 2 we introduce the new approach methodology that allows one to write down different finite, infinite, and infinitesimal numbers by a finite number of symbols as particular cases of a unique framework and to execute numerical computations with all of them. Then in Sec. 3 we briefly remind basic features of the percolation

phase transition and summarize some features of infinity percolation cluster in terms of infinity computations. In Sec. 4 we apply the new arithmetic to the cellular automaton forest-fire model. In the final section, the application results are summarized and discussed.

2 Methodology

In this section, we give a brief introduction to the new methodology that can be found in a rather comprehensive form in [27, 31, 35] downloadable from [24] (see also the monograph [22] written in a popular manner). A number of applications of the new approach can be found in [9, 10, 40, 25, 26, 28, 18, 19, 32, 33, 34, 35, 36, 37, 41]. We start by introducing three postulates that will fix our methodological positions (having a strong applied character) with respect to infinite and infinitesimal quantities and Mathematics, in general.

Usually, when mathematicians deal with infinite objects (sets or processes) it is supposed that human beings are able to execute certain operations infinitely many times. Since we live in a finite world and all human beings and/or computers finish operations they have started, this supposition is not adopted.

Postulate 1. *There exist infinite and infinitesimal objects but human beings and machines are able to execute only a finite number of operations.*

Due to this Postulate, we accept a priori that we shall never be able to give a complete description of infinite processes and sets due to our finite capabilities.

The second postulate follows the way of reasoning used in natural sciences where researchers use tools to describe the object of their study and the used instrument influences the results of observations. When physicists see a black dot in their microscope they cannot say: The object of observation *is* the black dot. They are obliged to say: the lens used in the microscope allows us to see the black dot and it is not possible to say anything more about the nature of the object of observation until we change the instrument – the lens or the microscope itself – by a more precise one.

Due to Postulate 1, the same happens in Mathematics studying natural phenomena, numbers, and objects that can be constructed by using numbers. Numeral systems used to express numbers are among the instruments of observations used by mathematicians. Usage of powerful numeral systems gives the possibility to obtain more precise results in Mathematics and in the same way usage of a good microscope gives the possibility of obtaining more precise results in Physics. However, the capabilities of the tools will be always limited due to Postulates 1 and 2, so, we shall never tell, **what is**, for example, a number but shall just observe it through numerals expressible in a chosen numeral system.

Postulate 2. *We shall not tell **what are** the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.*

Particularly, this means that from our point of view, axiomatic systems do not define mathematical objects but just determine formal rules for operating with certain numerals reflecting some properties of the studied mathematical objects. Throughout the paper, we shall always emphasize this philosophical triad – researcher, object of investigation, and tools used to observe the object – in various mathematical and computational contexts.

Finally, we adopt Euclid's Common Notion 5 'The whole is greater than the part' mentioned above as the third postulate.

Postulate 3. *The principle 'The part is less than the whole' is applied to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).*

Due to this declared applied statement, it becomes clear that the subject of this paper is out of Cantor's approach and, as a consequence, out of non-standard analysis of Robinson. Such concepts as bijection, numerable and continuum sets, cardinal and ordinal numbers cannot be used in this paper because they belong to the theory working with different assumptions. However, the approach used here does not contradict Cantor and Robinson. It can be viewed just as a more strong lens of a mathematical microscope that allows one to distinguish more objects and to work with them.

In [22, 27], a new numeral system has been developed in accordance with Postulates 1–3. It gives one a possibility to execute numerical computations not only with finite numbers but also with infinite and infinitesimal ones. The main idea consists of the possibility to measure infinite and infinitesimal quantities by different (infinite, finite, and infinitesimal) units of measure.

A new infinite unit of measure has been introduced for this purpose as the number of elements of the set \mathbb{N} of natural numbers. It is expressed by the numeral $\textcircled{1}$ called *grossone*. It is necessary to note immediately that $\textcircled{1}$ is neither Cantor's \aleph_0 nor ω . Particularly, it has both cardinal and ordinal properties as usual finite natural numbers (see [27]).

Formally, grossone is introduced as a new number by describing its properties postulated by the *Infinite Unit Axiom* (IUA) (see [22, 27]). This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integer numbers are introduced. It is important to emphasize that we speak about axioms of real numbers in sense of Postulate 2, i.e., axioms define formal rules of operations with numerals in a given numeral system.

Inasmuch as it has been postulated that grossone is a number, all other axioms for numbers hold for it, too. Particularly, associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers. This means that the following relations hold for grossone, as for any other number

$$0 \cdot \textcircled{1} = \textcircled{1} \cdot 0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \frac{\textcircled{1}}{\textcircled{1}} = 1, \quad \textcircled{1}^0 = 1, \quad 1^{\textcircled{1}} = 1, \quad 0^{\textcircled{1}} = 0. \quad (1)$$

Let us comment upon the nature of grossone by some illustrative examples.

Infinite numbers constructed using grossone can be interpreted in terms of the number of elements of infinite sets. For example, $\textcircled{1} - 1$ is the number of elements of a set $B = \mathbb{N} \setminus \{b\}$, $b \in \mathbb{N}$, and $\textcircled{1} + 1$ is the number of elements of a set $A = \mathbb{N} \cup \{a\}$, where $a \notin \mathbb{N}$. Due to Postulate 3, integer positive numbers that are larger than grossone do not belong to \mathbb{N} but also can be easily interpreted. For instance, $\textcircled{1}^2$ is the number of elements of the set V , where $V = \{(a_1, a_2) : a_1 \in \mathbb{N}, a_2 \in \mathbb{N}\}$.

Grossone has been introduced as the quantity of natural numbers. As a consequence, similarly to the set

$$A = \{1, 2, 3, 4, 5\} \quad (2)$$

consisting of 5 natural numbers where 5 is the largest number in A , $\textcircled{1}$ is the largest number¹ in \mathbb{N} and $\textcircled{1} \in \mathbb{N}$ analogously to the fact that 5 belongs to A . Thus, the set, \mathbb{N} , of natural numbers can be written in the form

$$\mathbb{N} = \{1, 2, \dots, \frac{\textcircled{1}}{2} - 2, \frac{\textcircled{1}}{2} - 1, \frac{\textcircled{1}}{2}, \frac{\textcircled{1}}{2} + 1, \frac{\textcircled{1}}{2} + 2, \dots, \textcircled{1} - 2, \textcircled{1} - 1, \textcircled{1}\}. \quad (3)$$

Note that traditional numeral systems did not allow us to see infinite natural numbers

$$\dots, \frac{\textcircled{1}}{2} - 2, \frac{\textcircled{1}}{2} - 1, \frac{\textcircled{1}}{2}, \frac{\textcircled{1}}{2} + 1, \frac{\textcircled{1}}{2} + 2, \dots, \textcircled{1} - 2, \textcircled{1} - 1, \textcircled{1}. \quad (4)$$

Similarly, Pirahã² are not able to see finite numbers larger than 2 using their weak numeral system but these numbers are visible if one uses a more powerful numeral system. Due to Postulate 2, the same object of observation – the set \mathbb{N} – can be observed by different instruments – numeral systems – with different accuracies allowing one to express more or less natural numbers. \square

This example illustrates also the fact that when we speak about sets (finite or infinite) it is necessary to take care about tools used to describe a set (remember Postulate 2). In order to introduce a set, it is necessary to have a language (e.g., a numeral system) allowing us to describe its elements and the number of the elements in the set. For instance, the set A from (2) cannot be defined using the mathematical language of Pirahã.

In order to express numbers having finite, infinite, and infinitesimal parts, records similar to traditional positional numeral systems can be used (see [22, 27]). To construct a number C in the new numeral positional system with base $\textcircled{1}$, we subdivide C into groups corresponding to powers of $\textcircled{1}$:

$$C = c_{p_m} \textcircled{1}^{p_m} + \dots + c_{p_1} \textcircled{1}^{p_1} + c_{p_0} \textcircled{1}^{p_0} + c_{p_{-1}} \textcircled{1}^{p_{-1}} + \dots + c_{p_{-k}} \textcircled{1}^{p_{-k}}. \quad (5)$$

Then, the record

$$C = c_{p_m} \textcircled{1}^{p_m} \dots c_{p_1} \textcircled{1}^{p_1} c_{p_0} \textcircled{1}^{p_0} c_{p_{-1}} \textcircled{1}^{p_{-1}} \dots c_{p_{-k}} \textcircled{1}^{p_{-k}} \quad (6)$$

represents the number C , where all numerals $c_i \neq 0$, they belong to a traditional numeral system and are called *grossdigits*. They express finite positive or negative numbers and show how many corresponding units $\textcircled{1}^{p_i}$ should be added or subtracted in order to form the number C .

Numbers p_i in (6) are sorted in the decreasing order with $p_0 = 0$

$$p_m > p_{m-1} > \dots > p_1 > p_0 > p_{-1} > \dots > p_{-(k-1)} > p_{-k}.$$

¹This fact is one of the important methodological differences with respect to non-standard analysis theories where it is supposed that infinite numbers do not belong to \mathbb{N} .

²Pirahã is a primitive tribe living in Amazonia that uses a very simple numeral system for counting: one, two, ‘many’ (see [16]). For Pirahã, all quantities larger than two are just ‘many’ and such operations as $2+2$ and $2+1$ give the same result, i.e., ‘many’. Using their weak numeral system Pirahã are not able to distinguish numbers larger than 2 and, as a result, to execute arithmetical operations with them. Another peculiarity of this numeral system is that ‘many’ + 1 = ‘many’. It can be immediately seen that this result is very similar to our traditional record $\infty + 1 = \infty$.

They are called *grosspowers* and they themselves can be written in the form (6). In the record (6), we write $\textcircled{1}^{p_i}$ explicitly because in the new numeral positional system the number i in general is not equal to the grosspower p_i . This gives the possibility to write down numerals without indicating grossdigits equal to zero.

The term having $p_0 = 0$ represents the finite part of C because, due to (1), we have $c_0 \textcircled{1}^0 = c_0$. The terms having finite positive grosspowers represent the simplest infinite parts of C . Analogously, terms having negative finite grosspowers represent the simplest infinitesimal parts of C . For instance, the number $\textcircled{1}^{-1} = \frac{1}{\textcircled{1}}$ is infinitesimal. It is the inverse element with respect to multiplication for $\textcircled{1}$:

$$\textcircled{1}^{-1} \cdot \textcircled{1} = \textcircled{1} \cdot \textcircled{1}^{-1} = 1. \quad (7)$$

Note that all infinitesimals are not equal to zero. Particularly, $\frac{1}{\textcircled{1}} > 0$ because it is a result of division of two positive numbers. All of the numbers introduced above can be grosspowers, as well, giving thus a possibility to have various combinations of quantities and to construct terms having a more complex structure.

3 Geometric phase transition

Consider a d -dimensional hypercubic lattice, where each site is either occupied randomly with a probability p or empty with probability $1 - p$. Occupied and empty sites may stand for very different physical properties [4, 5, 15, 39]. For purposes of further inquiry, let us assume that the occupied sites are trees, and that combustion front can flow only between nearest neighbor occupied sites. At a low concentration p , the occupied sites are either isolated or form small clusters of nearest neighbor trees. We suppose that two occupied sites belong to the same cluster if they are connected by a path of nearest neighbor occupied sites, and a fire can flow between them. At low values of p only small clusters of occupied sites exist. When the concentration p increases, the average size of the clusters increases, as well. At the critical concentration p_c , a large cluster appears which connects opposite edges of the lattice. This cluster is commonly named the *spanning cluster* or *percolating cluster* [4, 5, 15, 39]. In the thermodynamic limit, i.e. in the infinite system limit spanning cluster named *infinite cluster*, since its size diverges when the size of the lattice increases to infinity. It should be emphasized here that from a traditional standpoint there exists a unique *infinite* cluster and this *infinite* cluster always coincides with the *spanning* cluster [4, 5, 15, 39].

In contrast to the more common thermal phase transitions, where the transition between two phases occurs at a critical temperature, the percolation transition described here is a geometrical phase transition, which is characterized by the geometric features of large clusters in the neighborhood of p_c .

When p increases further, the density of the infinite cluster also increases, since more and more sites start to be a part of the infinite cluster. Simultaneously, the average size of the finite clusters, which do not belong to the infinite cluster, decreases. At $p = 1$, trivially, all sites belong to the infinite cluster.

In percolation, the concentration p of occupied sites plays the same role as the temperature in thermal phase transitions. Similar to thermal transitions, long range correlations control

the percolation transition and the relevant quantities near p_c are described by power laws and critical exponents [4, 5, 15, 39].

The percolation transition is characterized by the geometrical properties of clusters for values of p that are close to p_c . One of important characteristics describing these properties is the probability, P_∞ , that a site belongs to the infinite cluster. For $p < p_c$, only finite clusters exist, and, therefore, it follows $P_\infty = 0$. For values $p > p_c$, P_∞ behaves similarly to the magnetization below critical temperature, and increases with p by a power law

$$P_\infty \sim (p - p_c)^\beta, \quad (8)$$

where β is critical exponent of the order parameter [4, 5, 15, 39].

The linear size of the finite clusters, below and above percolation transition, is characterized by the *correlation length* ξ . The correlation length is defined as the mean distance between two sites on the same finite cluster. When p approaches p_c , ξ increases as

$$\xi \simeq a \cdot |p - p_c|^{-\nu}, \quad (9)$$

with the same exponent ν below and above the threshold [4, 5, 15, 39], and with a lattice spacing a .

The structure of percolation cluster can be well described in the framework of the fractal theory. We begin by considering the percolation cluster at the critical concentration p_c . The cluster is self-similar on all length scales (larger than the unit size and smaller than the lattice size), and can be regarded as a fractal. The *fractal dimension*, d_f , describes how, on the average, the mass, M , of the cluster within a sphere of radius r scales with the r , [4, 5, 15]

$$M(r) \sim r^{d_f}. \quad (10)$$

In random fractals, $M(r)$ represents an average over many different cluster configurations or, equivalently, over many different centers of spheres on the same *infinite* cluster. Below and above p_c , the mean size of the finite clusters in the system is described by the correlation length ξ . At p_c , ξ diverges and holes occur in the *infinite* cluster on all length scales. Above p_c , ξ also represents the linear size of the holes in the *infinite* cluster. Since ξ is finite above p_c , the *infinite* cluster can be self-similar only on length scales smaller than ξ . We can interpret $\xi(p)$ as a typical length up to which the cluster is self-similar and can be regarded as a fractal. For length scales larger than ξ , the structure is not self-similar and can be regarded as homogeneous. It is worth to mention that the correlation length ξ can be also regarded as a cut-off bound of a coarse graining analysis. A fine-grained description of a system is a detailed, low-level model of it. A coarse-grained description is a model where some of this fine detail has been smoothed over or averaged out. The replacement of a fine-grained description with a lower-resolution coarse-grained model is called coarse graining [20]. Indeed, the end to which our percolation system may be coarse grained to keep its fractal properties is simply bound by the length ξ .

One can relate the fractal dimension d_f of percolation cluster to the exponents β and ν [4, 5, 15, 39]:

$$d_f = d - \frac{\beta}{\nu}. \quad (11)$$

Thus the fractal dimension of the *infinite* cluster at p_c is not a new independent exponent but depends on β and ν . Since β and ν are universal exponents, d_f is also universal. It can be

shown [39] that (11) also represents the fractal dimension of the finite clusters at p_c and below p_c , as long as their linear size is smaller than ξ .

The fractal dimension, however, is not sufficient to fully characterize a percolation cluster. For a further intrinsic characterization of a fractal we consider the shortest path between two sites on the cluster. We denote the length of this path, which is called the ‘chemical distance’, by l [4, 5, 15, 39]. The *graph* dimension d_l , which is also called the ‘chemical’ or ‘topological’ dimension, describes how the cluster mass M within the chemical distance l from a given site scales with l [4, 5],

$$M(l) \sim l^{d_l}. \quad (12)$$

While the fractal dimension d_f characterizes how the mass of the cluster scales with the Euclidean distance r , the graph dimension d_l characterizes how the mass scales with the chemical distance l . To characterise the chemical distance l from a given site scales with the Euclidean distance r one typically uses the shortest pass dimension d_{min} on percolation cluster [4, 5]

$$d_{min} = \frac{d_f}{d_l}. \quad (13)$$

The concept of the chemical distance and the shortest pass dimension also plays an important role in the description of spreading phenomena such as epidemics and forest fires, which propagate along the shortest pathes from the seed.

Recently we considered the percolation phase transition by means of new computational methodology [2]. It has been established that in an infinite system phase transition point is not really a point as with respect of traditional approach. In light of new arithmetic it appears as a critical interval, rather than a critical point. Depending on the “microscope” we use this interval could be regarded as either finite or infinite or infinitesimal interval. Using the new approach we observed that in vicinity of percolation threshold we have *many* different *infinite clusters* instead of *one infinite cluster* that appears in traditional consideration. Moreover, we have now a tool to distinguish those infinite clusters. In particular, we can distinguish *spanning infinite clusters* from *embedded infinite clusters*.

4 Forest Fires Model

In this section, we are going to apply the new arithmetic to a self-organized critical forest-fire model (see [12, 13]) which is tightly connected with the percolation methodology and in some sense combines the dynamic and the static percolation problems. Let us assume, that we examine the forest fire model on a d -dimensional hypercubic lattice with the lattice spacing a and on the linear scale $L = a \cdot \textcircled{1}$. Then, the lattice contains the infinite number of sites $N = \textcircled{1}^d$. A lattice site can be in one of the following three states: empty, a tree, or a burning tree. The forest-fire model is a stochastic cellular automaton in which a configuration at every time step evolves according to the following rules [3]:

1. A tree grows in an empty site with a probability p ;
2. A site with a burning tree becomes an empty site;

3. A tree becomes a burning tree if at least one of its neighbors is a burning tree;
4. A tree without burning neighbors catches fire spontaneously with a probability ε .

Starting with arbitrary initial conditions, the system approaches after a transition period a steady state the properties of which depend only on the parameter values. Let ρ_e , ρ_t , and ρ_f be the mean overall density of empty sites, of trees, and of burning trees in the system in the steady state, respectively. These densities are related by the equations

$$\rho_e + \rho_t + \rho_f = 1 \quad (14)$$

and

$$p\rho_e = \rho_f. \quad (15)$$

The last relation says that the mean number of growing trees equals the mean number of burning trees in the steady state [14].

During one time step, there are

$$\varepsilon\rho_t N \quad (16)$$

lightning strokes in the system and

$$p\rho_e N \quad (17)$$

growing trees. Therefore in the steady state the mean number \bar{s} of trees that are destroyed by a lightning stroke is

$$\bar{s} = p\rho_e N (\varepsilon\rho_t N)^{-1} = \frac{p}{\varepsilon} \frac{1 - \rho_t - \rho_f}{\rho_t} \quad (18)$$

When the fire density is large, trees cannot live long enough to become a part of large forest clusters. So, large-scale structures we are interested in can only occur when the fire density ρ_f is small. Equation (15) shows that the fire density becomes small when the tree growth rate p approaches zero. Therefore value p should be represented as an infinitesimal small number, say $p = \varepsilon^{-\phi}$, where $\phi > 0$. When $p \ll 1$ and consequently in compliance with (15) $\rho_f \ll \rho_e$ one could rewrite the last equation in the following approximate form [12, 14]

$$\bar{s} = p\rho_e N (\varepsilon\rho_t N)^{-1} = \frac{p}{\varepsilon} \frac{1 - \rho_t}{\rho_t}. \quad (19)$$

As it was already mentioned above, in the model discussed the forest fire can also be considered as a percolation process [17]. This description is ‘mean field’ in the sense that spatial correlations are neglected and trees are considered as uniformly randomly distributed over the lattice at a density equal to ρ_t . A cluster is defined as a set of trees that are all connected through nearest neighbor links. If one tree in the cluster catches fire then the whole cluster will eventually burn down, because the fire will be able to spread through nearest neighbor links to all trees in the cluster. This way of considering the forest fire model allows one to make use of several results from the percolation theory. First of all, in the mean field description $\rho_t \lesssim x_c$, where $x_c \leq 1$ is the d -dimensional percolation threshold, for example, $x_c \simeq 0.59$ and $x_c \simeq 0.35$ in 2D and 3D site percolation problem correspondently. Therefore, the second factor of the right-hand side of Eq.(19) is of the order of one [12, 14], and Eq.(19) then represents a power law [12, 14]

$$\bar{s} \sim \left(\frac{\varepsilon}{p} \right)^{-1}. \quad (20)$$

Moreover, we have to choose the tree growth rate p so small that even the largest forest cluster burns down rapidly, before new trees grow up at its edge. The last statement implies that

$$p \ll T_L^{-1}, \quad (21)$$

where T_L is the time the fire needs to burn down. In addition, the lightning probability ε must satisfy

$$\varepsilon \ll p \quad (22)$$

Otherwise, a tree is destroyed by lightning before its neighbors grow up, and no large-scale structures can be formed. The inequalities (21) and (22) represent a double separation of time scales

$$T_L \ll p^{-1} \ll \varepsilon^{-1}, \quad (23)$$

which is the condition for self-similar behavior in the forest fire model. In this case, the dynamics of the system depends only on the ratio ε/p , but not on ε and p separately. The values ε , and ε/p also could be represented as infinitesimal small numbers. The most essential is the ratio ε/p , and for the beginning we choose $\varepsilon/p \simeq \mathbb{1}^{-\theta}$, where $\theta > 0$ is a finite number. So, we can rewrite expression (20) as following

$$\bar{s} \sim \mathbb{1}^\theta. \quad (24)$$

The mean number \bar{s} of trees that are destroyed by a lightning stroke is obviously less than $N = \mathbb{1}^d$. Therefore in the steady state it follows that

$$\theta < d. \quad (25)$$

Lightning will strike the system every $T_f = (\varepsilon \rho_t N)^{-1}$ time steps on average. In order to obtain nontrivial dynamics we should keep the lightning waiting time T_f short enough in comparison with time interval p^{-1} that system requests to be overgrown with forest trees. Otherwise the model discussed will demonstrate saw-tooth overall forest density oscillations: firstly, the model grid is overgrown with forest trees, then one lightning stroke sets fire to the bush, completely destroying the forest and prepare a place for the new generation. Moreover, increasing the system size in order to prevent the fire from dying out we observe the fire fronts in the form of more or less smooth and regular spirals. These spirals represent self-sustained dissipative structures or combustion autowaves. The characteristic spatial scale of the autowaves, is of the order of $a \cdot p^{-1}$ [11]. Thus, in order to obtain self similar forest fires dynamics we have to fulfil the following condition

$$T_f = (\varepsilon \rho_t N)^{-1} \ll p^{-1} \quad (26)$$

and again so long as $N = \mathbb{1}^d$ and $\varepsilon/p \simeq \mathbb{1}^{-\theta}$ we have to satisfy the following inequality

$$\theta < d. \quad (27)$$

Mean field consideration of the forest fire model allows us to make use of a couple more results from percolation theory. The first is the result that the number of trees in a forest cluster is simply $(R/a)^{d_f}$ where d_f is the percolation cluster fractal dimension and R is the cluster gyration radius (see (10)). Then the largest forest cluster with gyration radius of the order of

$R \sim L = a \cdot \mathbb{1}$ contains approximately $(L/a)^{d_f} = \mathbb{1}^{d_f}$ elements. The number of the burnt out trees (the size of a fire) is simply the number of destroyed cluster elements. For the largest fire, this number under the order of size will make $(L/a)^{d_f} = \mathbb{1}^{d_f}$. So, instead of (27) we have to satisfy the following inequality

$$\theta < d_f. \quad (28)$$

The second is the result that the time T_R the fire needs to burn down a forest cluster of size R is determined by the shortest pass dimension d_{min} on percolation cluster

$$T_R = \left(\frac{R}{a}\right)^{d_{min}}. \quad (29)$$

The scale of the largest fire which can arise on our lattice is comparable to lattice size $R \sim L = a \cdot \mathbb{1}$. Thus, the maximal forest fire duration will make

$$T_L \simeq \left(\frac{L}{a}\right)^{d_{min}} \simeq \mathbb{1}^{d_{min}}. \quad (30)$$

Let us introduce the observer of our system, that uses a new time scale with step equal to the maximal forest fire duration, i.e. $\mathbb{1}^{d_{min}}$ steps of initial modelling time. For such an observer even the largest forest cluster is burned down instantaneously, i.e., during one time step when one of its trees is struck by lightning. A typical example of the forest fire model time evolution that our observer could observe is shown in Fig. 1. The top panel of the figure represents burning

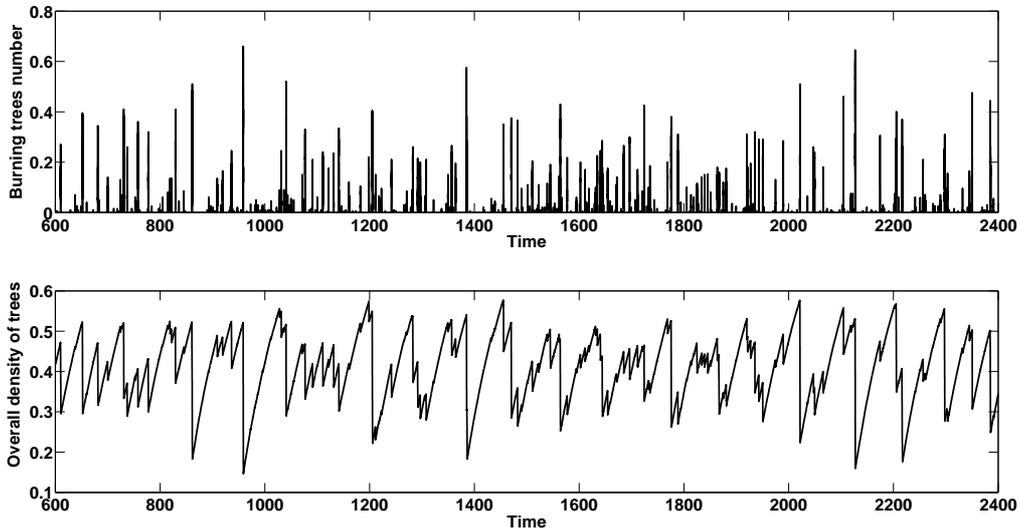


Figure 1: The forest fire model time evolution in two dimensions. Time scales with steps equal to T_L

tree number time evolution. Time scales with step equals to T_L . The ordinate axis tick labels should be multiplied by $\frac{L}{a}$ factor. One may choose an extremely huge value of $\frac{L}{a}$, even $\frac{L}{a} = \mathbb{1}$. The bottom panel of the figure represents time evolution of the overall tree density. From the

point of view of our observer the overall tree density slumps instantly during forest fire sparks. The tree growth rate p and lightning probability ε for such an observer will be changed by the following values

$$\dot{p} = p \cdot T_L, \quad \dot{\varepsilon} = \varepsilon \cdot T_L, \quad (31)$$

and can be represented by the infinitesimal numbers

$$\dot{p} \simeq \mathbb{1}^{d_{min}-\phi}, \quad \dot{\varepsilon} \simeq \mathbb{1}^{d_{min}-\phi-\theta}. \quad (32)$$

The first expression in (32) represents an infinitesimal number when

$$\phi > d_{min}, \quad (33)$$

in compliance with inequality (21) and estimation (30).

From the point of view of our observer the system time evolution looks like a set of sparks of different intensity (see Fig.2 that just zooms up an image patch in Fig.1). One separate spark

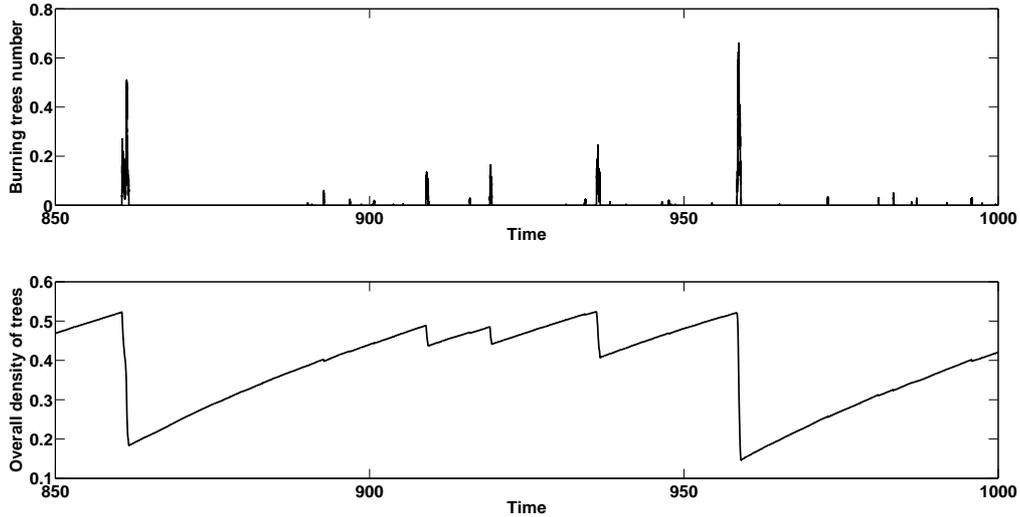


Figure 2: The forest fire model time evolution in two dimensions. Zoom of image patch in Fig.1. Time scales with step equals to T_L

could be represented as a product of discrete delta-functions (one step – one tree in fire) and the number of trees in a forest cluster that catches the fire. But when we are going to investigate the forest fire internal structure, its inherent dynamic, our observer could not help us. In the case we have to use more powerful ”microscope”. The situation is depicted in Fig.3 that in its turn zooms up an image patch in Fig.2. In the figure time makes only a couple of steps each equals to $\mathbb{1}^{d_{min}}$. To launch a narrow analysis of fire spreading we should choose the time resolution that is considerably less than T_L . The model with instantaneous combustion is referred to as a simplified version of the real forest-fire model [12, 14]. Now one can see that we deal with the same model but with different time resolution.

Large-scale structures and therefore criticality can only occur when different forest fires or different sparks are well separated from each other. Otherwise the system will undergo exposure

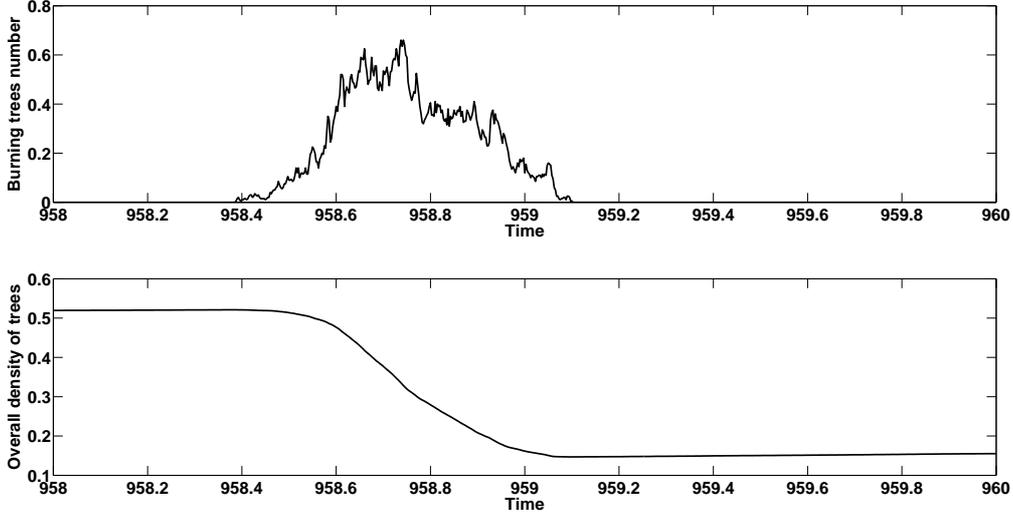


Figure 3: The forest fire model time evolution in two dimensions. Zoom of image patch in Fig.2. Time scales in T_L units

to several fires simultaneously. This separation is well reflected in Fig.2. Different forest fires in our system are well separated in time when following condition is fulfilled

$$\dot{\varepsilon} \cdot N \rho_t \sim \mathbb{1}^{d+d_{min}-\theta-\phi} \ll 1. \quad (34)$$

It means that

$$\theta + \phi > d + d_{min}. \quad (35)$$

The last inequality combined with inequality (28) provide us with more rigid condition in comparison with inequality (33)

$$\phi > d_{min} + (d - d_f) \quad (36)$$

as far as in any dimension $d > d_f$.

Thus, to observe criticality and large-scale structures we have to fulfil the following system of inequalities

$$\begin{cases} \phi > d_{min} + (d - d_f) \\ \theta < d_f \\ \theta + \phi > d + d_{min} \end{cases} . \quad (37)$$

Infinite and infinitesimal numbers introduced in [22, 27, 31] allow us to decide on the order of priorities in our model space. When we determine the system linear size $L = a \cdot \mathbb{1}^1$ the following choice of the model parameters waits on the results of the choice of the exponents ϕ and θ

$$p = \mathbb{1}^{-\phi}; \quad \varepsilon = \mathbb{1}^{-(\phi+\theta)}. \quad (38)$$

In the 2D case, for example, we can use the following parameter values in concordance with (37)

$$\phi = 2; \quad \theta = \frac{3}{2}, \quad (39)$$

and consequently

$$p = \mathbb{1}^{-2}; \quad \varepsilon = \mathbb{1}^{-3.5}. \quad (40)$$

5 A brief conclusion

In this paper, it has been shown that infinite and infinitesimal numbers introduced in [22, 27, 31] allow us to obtain exact numerical results instead of traditional asymptotic forms at different points at infinity. We consider a number of traditional models related to the percolation theory using the new computational methodology. It has been shown that the new computational tools allow one to create new, more precise models of percolation and to study the existing models more in detail. The introduction in these models new, computationally manageable notions of the infinity and infinitesimals gives a possibility to pass from the traditional qualitative analysis of the situations related to these values to the quantitative one. Naturally, such a transition is very important from both theoretical and practical viewpoints.

The point of view presented in this paper uses strongly two methodological ideas borrowed from Physics: relativity and interrelations holding between the object of an observation and the tool used for this observation. The latter is directly related to connections between Analysis and Numerical Analysis because the numeral systems we use to write down numbers, functions, etc. are among our tools of investigation and, as a result, they strongly influence our capabilities to study mathematical objects.

Scientists that deal with forest-fire model and its applications, distinguish between two versions of the model: real forest-fire model where fire catches adjacent trees in the forest in the step by step manner and simplified version with instantaneous combustion [12, 14]. Using the new approach we show that in both situations we deal with the same model but with different time resolution. We observe that depending on the “microscope” we use the same cellular automaton forest-fire model reveals either instantaneous forest combustion or step by step firing. By means of the new approach it was also observed that the scaling properties of the system to be very sensitive to the trees growing rate and to the ratio between the ignition probability and the growth probability. As far as we choose infinitesimal values of the two parameters we immediately determine the measure or extent of the system size infinity that provides the criticality of the system dynamics. Correspondent inequalities for grosspowers are derived.

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