Letter

Letter to the Editor

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The Mathematical Intelligencer *encourages comments about the material in its issues. Letters to the editor should be sent to the editor-in-chief.*

The Editor-in-Chief of *The Mathematical Intelligencer*, Prof. Marjorie Senechal, has invited me to answer a letter [1] concerning my recent note *The Olympic Medals Ranks, Lexicographic Ordering, and Numerical Infinities* [2]. I am always happy to communicate with readers and thank Prof. Senechal for giving me this opportunity to reply.

The reader writes that results presented in [2] can be obtained "in many different ways (say, with positional notation for ordinals, or using non-Archimedean fields and nonstandard models for reals)," but he does not suggest that this can be done *numerically*. I have explicitly stated (see [2], page 5, before formula (5)) that these computations can be executed symbolically and, in fact, all the ways the reader lists of dealing with infinity are *symbolic*, whereas *numerical infinities* are the most important words in my title. Numerical computations work with approximate floating point numbers, whereas symbolic computations are the exact manipulations with mathematical expressions containing *variables with no given value* and are thus manipulated as symbols.

Let me explain this point in more detail, emphasizing also some differences between the ①-based methodology and traditional approaches to infinity and infinitesimals. Let us start with analyzing numerical computations with finite quantities. When we execute these computations, the *same* numerals are used for *different* purposes (e.g., 10 can express the number of elements of a set, indicate the position of an element in a sequence, or execute practical numerical computations). In contrast, when we face the necessity of working with infinities and/or infinitesimals, the situation changes drastically and we face a number of distinctions and complications.

First, *different* numerals are used to work with infinities and infinitesimals in different situations. For example we use the symbol ∞ in standard analysis, symbol ω for working with ordinals, symbols $\aleph_0, \aleph_1, \ldots$ for dealing with cardinalities, etc.

Second, traditionally theories dealing with infinite and infinitesimal quantities have a symbolic (not numerical) character and only algebraic manipulations can be done. For instance, nonstandard models and non-Archimedean fields use either a *generic* infinite number or a *generic* infinitesimal in their constructions (e.g., Levi-Civita numbers are built using a generic infinitesimal ε), whereas our numerical computations with finite quantities are concrete and not generic. If we consider a finite n, then different values can be assigned to it, for example, we can use the numeral 34 and write $n = 34$. Clearly, any other numeral used to express finite quantities and consisting of a finite number of symbols can be taken for this purpose. The finiteness of the number of symbols is necessary for executing practical computations, because we should be able to write down (and/or store) values with which we execute operations.

In contrast, if we consider a nonstandard infinite m then it is not clear which numerals consisting of a finite number of symbols can be used to assign a concrete value to m. Again, it is not clear which numerals can be used to assign a value to the generic infinitesimal ε and to write $\varepsilon = \ldots$. Moreover, approaches of this kind leave unclear such issues as, for example, whether the infinite $1/\varepsilon$ is an integer or not, and whether $1/\varepsilon$ is the number of elements of a concrete infinite set. If one wishes to consider two infinitesimals (or infinities) h_1 and h_2, where h_2 is not expressed in terms of h_1, then it is not clear how to compare them because numeral systems that can express different values of infinities and infinitesimals are not provided by this kind of technique. In contrast, when we work with finite quantities, then we can compare n and k if they assume numerical values, for example, if $k = 25$ and $n = 78$, then, by using rules of the numeral system the symbols 25 and 78 belong to, we can compute that $n > k$.

Third, many arithmetics used to deal with infinities not only should be used for specific purposes only, but in addition are quite different with respect to the way we execute computations with finite quantities. Let me give some examples:

- There exist undetermined operations ($\infty - \infty$, $\frac{\infty}{\infty}$, etc.) that are absent when we work with finite numbers.
- Arithmetic with ordinals is very different from arithmetic with finite quantities. For instance, addition and multiplication are not commutative (e.g., $1 + \omega = \omega$ and $\omega + 1 > \omega$), there does not exist an ordinal γ such that $\gamma + 1 = \omega$, etc.
- Arithmetic with ∞ and infinite cardinals does not satisfy Euclid's Common Notion 5 saying "The whole is greater than the part" that holds when we work with finite

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quantities. For example, $\infty + 1 = \infty$ and $\aleph_0 + 1 = \aleph_0$ whereas $x + 1 > x$ for any finite x.

The difference between my computational methodology and the traditional approaches is that mine is free of these complications and allows one to work very easily with *concrete* infinities and infinitesimals *numerically*. It applies *the same numerals* in *all* the situations requiring infinities and infinitesimals just as when we work with numerals expressing finite quantities. It does not require a knowledge of cardinals, ordinals, ultrafilters, standard and nonstandard numbers, internal and external sets, etc. As the Italian logician Prof. Lolli has written [3], "This is simpler than nonstandard enlargements in its conception, it does not require infinitistic constructions and affords easier and stronger computation power." Moreover, in certain cases it provides new results for old problems and these results have a higher accuracy than with traditional tools. For instance,

– The same numerals can be used for working with functions and their derivatives that can assume different infinite, finite, and infinitesimal values and can be defined over infinite and infinitesimal domains. The notions of continuity and differentiability can be introduced not only for functions assuming finite values but for functions assuming infinite and infinitesimal values, as well.
– Divergent series do not exist and, in general, series are substituted by sums having a concrete infinite number of addends and, for a different number of addends, results (that can assume different infinite, finite, or infinitesimal values) are different as it happens for sums with a finite number of summands.
– There are no divergent integrals, limits of integration can be concrete different infinite, finite, or infinitesimal numbers, and results can assume different infinite, finite, or infinitesimal values.

In my note [2], I gave an example of numerical computations with ①-based numerals. I remark here only that ①-based numerals allow us to express the exact number of elements of certain infinite sets. The new numeral system and the numeral system of infinite cardinals do not contradict one another: both numeral systems provide correct answers, but their answers have *different accuracies*. As an analogy with physics, we can say that the lens of our new "telescope," used to observe infinities (and infinitesimals), is stronger, and where Cantor's "telescope" allows one to distinguish just two dots (countable sets and the continuum), we are able to see many different dots (infinite sets having different numbers of elements).

There is more to say, but this suffices for a brief letter. For further details, reviews, and applications related to numerical differentiation and optimization, ODEs, fractals, cellular automata, Euclidean and hyperbolic geometry, percolation, infinite series and the Riemann zeta function, set theory and the first Hilbert problem, Turing machines, etc., see http://www.theinfinitycomputer.com.

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REFERENCES

[1] A. Shen, Letter ''Nonsense,'' *The Mathematical Intelligencer*, 2015, 37(4).

[2] Ya. D. Sergeyev, ''The Olympic Medals Ranks, Lexicographic Ordering, and Numerical Infinities,'' *The Mathematical Intelligencer*, 2015, 37(2), 4–8.

[3] G. Lolli, ''Infinitesimals and Infinites in the History of Mathematics: A Brief Survey,'' *Applied Mathematics and Computation*, 2012, 218(16), 7979–7988.