# Generation of symmetric exponential sums

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**Abstract**

In this paper, a new method for generation of infinite series of symmetric identities written for exponential sums in real numbers is proposed. Such systems have numerous applications in theory of numbers, chaos theory, algorithmic complexity, dynamic systems, etc. Properties of generated identities are studied. Relations of the introduced method for generation of symmetric exponential sums to the Morse-Hedlund sequence and to the theory of magic squares are established.

**Key Words**: Real exponential sums, generation, Morse-Hedlund sequences, magic squares.

## 1 Introduction

Let us consider two sets of numbers, $\{x_i : 1 \leq i \leq k\}$ and $\{y_i : 1 \leq i \leq k\}$ such that the following system of identities is satisfied.

$$\left\{ \begin{array}{lcl} x_1^1 + x_2^1 + x_3^1 + \ldots + x_{k-1}^1 + x_k^1 & = & y_1^1 + y_2^1 + y_3^1 + \ldots + y_{k-1}^1 + y_k^1 \\ x_1^2 + x_2^2 + x_3^2 + \ldots + x_{k-1}^2 + x_k^2 & = & y_1^2 + y_2^2 + y_3^2 + \ldots + y_{k-1}^2 + y_k^2 \\ x_1^3 + x_2^3 + x_3^3 + \ldots + x_{k-1}^3 + x_k^3 & = & y_1^3 + y_2^3 + y_3^3 + \ldots + y_{k-1}^3 + y_k^3 \\ & \ldots\ldots\ldots & \\ x_1^n + x_2^n + x_3^n + \ldots + x_{k-1}^n + x_k^n & = & y_1^n + y_2^n + y_3^n + \ldots + y_{k-1}^n + y_k^n \end{array} \right. \tag{1}$$

Systems of the type (1) have a significant importance in mathematics. Such systems were widely studied in the number theory (see [4, 13, 18, 23]). For example, the Tarry-Escott problem (see, for example, [2, 17]) deals with finding disjoint sets $\{x_i : 1 \leq i \leq k\}$ and as $\{y_i : 1 \leq i \leq k\}$ of integers satisfying (1). The systems (1) have also a straightforward connection to the Hilbert-Kamke problem for integers (see [8, 10, 11]) being an extension of the famous Waring problem (see [24]).

Identities (1) considered as a description of the process of exponential sums expansion have close relations to the chaos theory (see [16]) and its applications, for instance, in economics (see [5]). Integer solutions to (1) in the context of integrals of piece-wise constant functions, emergent calculations, and integer code series and shows that identities (1) can be viewed (see [14, 15]) in terms of structural complexity of systems and algorithms. Particularly, studying structural complexity of multiextremal functions gives new tools for solving global optimization problems being a new actively raising research area (see [9, 14, 15, 22]).

Together with the system (1) the following system (2) can be considered.

$$\begin{cases} x_1^1+x_2^1+x_3^1+\ldots+x_{\theta_1-1}^1+x_{\theta_1}^1 & = & y_1^1+y_2^1+y_3^1+\ldots+y_{\theta_1-1}^1+y_{\theta_1}^1 \\ x_1^2+x_2^2+x_3^2+\ldots+x_{\theta_2-1}^2+x_{\theta_2}^2 & = & y_1^2+y_2^2+y_3^2+\ldots+y_{\theta_2-1}^2+y_{\theta_2}^2 \\ x_1^3+x_2^3+x_3^3+\ldots+x_{\theta_3-1}^3+x_{\theta_3}^3 & = & y_1^3+y_2^3+y_3^3+\ldots+y_{\theta_3-1}^3+y_{\theta_3}^3 \\ & \cdots\cdots\cdots & \\ x_1^n+x_2^n+x_3^n+\ldots+x_{k-1}^n+x_k^n & = & y_1^n+y_2^n+y_3^n+\ldots+y_{k-1}^n+y_k^n \end{cases} \tag{2}$$

Such systems have interesting connections (see [15]) to algorithmic complexity (see [6, 12, 21]) and period doubling in physical systems (see [7, 15]). In this case, the numbers $\theta_1, \theta_2, \ldots, \theta_{n-1}, \theta_n = k$ from (1) become

$$\theta_2 = 2\theta_1, \quad \theta_3 = 2\theta_2, \quad \ldots, \quad \theta_n = 2\theta_{n-1} \tag{3}$$

and, for example, for $\theta_1 = 2$ the following piramide is obtained.

$$\begin{array}{rcl} x_1^1+x_2^1 & = & y_1^1+y_2^1 \\ x_1^2+x_2^2+x_3^2+x_4^2 & = & y_1^2+y_2^2+y_3^2+y_4^2 \\ x_1^3+x_2^3+x_3^3+\ldots+x_7^3+x_8^3 & = & y_1^3+y_2^3+y_3^3+\ldots+y_7^3+y_8^3 \\ & \cdots\cdots\cdots & \\ x_1^n+x_2^n+x_3^n+\ldots+x_{k-1}^n+x_{2^n}^n & = & y_1^n+y_2^n+y_3^n+\ldots+y_{k-1}^n+y_{2^n}^n \end{array} \tag{4}$$

In this paper, for any integer power $n$, a method for generation of two sets $\{x_i : 1 \le i \le k\}$ and $\{y_i : 1 \le i \le k\}$ of real numbers satisfying both (1) and (2) for certain numbers $\theta_1 \le \theta_2 \le \ldots, \theta_{n-1} \le \theta_n = k$ is proposed. Particularly, piramidale systems of the type (2), (3) are considered. Properties of the introduced sets $\{x_i : 1 \le i \le k\}$ and $\{y_i : 1 \le i \le k\}$ are studied. It is shown that the sets found for a given $n$ are a basis for the sets satisfying (1) and (2) with the powers $n+1, n+2, \ldots$. In order to obtain solutions to a system having a higher power $n+1$ it is necessary to add certain numbers to the sets $\{x_i : 1 \le i \le k\}$ and $\{y_i : 1 \le i \le k\}$ found for the power $n$. In such a way, by increasing $n$, it is possible to speak about sequences $\{x_i\}$ and $\{y_i\}$ satisfying a sequence of systems (1) and (2).

The next section contains the main results. A few examples are given in the last section. It also presents relations of the introduced method for generation symmetric exponential sums to the Morse-Hedlund sequences (see [19]) being a particular case of the Prouhet sequences (see [20]) and to the theory of magic cubes developed in [1, 2].

## 2 Main results

In this section, two sets of real numbers $\{x_i : 1 \le i \le k\}$ and $\{y_i : 1 \le i \le k\}$ satisfying (1) and (2) for any given power $n$ are constructed[1]. Let us start by choosing four real numbers $a, b, c$, and $d$ satisfying the following simple equality

$$a + b = c + d. \tag{5}$$

Given numbers $a, b, c, d$ from (5) and any real numbers $k_1, k_2, \dots, k_n, n \ge 2$, let us introduce by recursion functions $L^j(k_1, k_2, \dots, k_{n-1})$ and $R^j(k_1, k_2, \dots, k_{n-1})$, where $j = n-1, n$. The functions $L^1(k_1), R^1(k_1), L^2(k_1), R^2(k_1)$ and $L^2(k_1, k_2), R^2(k_1, k_2)$ are defined as follows:

$$L^1(k_1) = (a + k_1)^1 + (b + k_1)^1 = x_1^1 + x_2^1, \tag{6}$$

where $x_1^1 = a + k_1$ and $x_2^1 = b + k_1$,

$$R^1(k_1) = (c + k_1)^1 + (d + k_1)^1 = y_1^1 + y_2^1, \tag{7}$$

where $y_1^1 = c + k_1$ and $y_2^1 = d + k_1$,

$$L^2(k_1) = (a + k_1)^2 + (b + k_1)^2 = x_1^2 + x_2^2,$$

$$R^2(k_1) = (c + k_1)^2 + (d + k_1)^2 = y_1^2 + y_2^2.$$

Then, setting $x_3 = c + k_1 + k_2, x_4 = c + k_1 + k_2$, and $R^2(k_1 + k_2) = x_3^2 + x_4^2$ we define

$$L^2(k_1, k_2) = L^2(k_1) + R^2(k_1 + k_2) = x_1^2 + x_2^2 + x_3^2 + x_4^2 =$$

$$(a + k_1)^2 + (b + k_1)^2 + (c + k_1 + k_2)^2 + (d + k_1 + k_2)^2.$$

Analogously we introduce $y_3 = a + k_1 + k_2, y_4 = b + k_1 + k_2$, and $L^2(k_1 + k_2) = y_3^2 + y_4^2$ to define

$$R^2(k_1, k_2) = R^2(k_1) + L^2(k_1 + k_2) = y_1^2 + y_2^2 + y_3^2 + y_4^2 =$$

$$(c + k_1)^2 + (d + k_1)^2 + (a + k_1 + k_2)^2 + (b + k_1 + k_2)^2.$$

Now, using already determined functions, we can introduce by recursion functions $L^j(k_1, k_2, \dots, k_{n-1})$ where $j = n-1, n$:

$$L^n(k_1, k_2, \dots, k_{n-1}, k_n) = L^n(k_1, k_2, \dots, k_{n-1}) + R^n(k_1 + k_n, k_2, \dots, k_{n-1}) = \sum_{i=1}^{2^n} x_i^n, \tag{8}$$

where

$$L^n(k_1, k_2, \dots, k_{n-1}) = \sum_{i=1}^{2^{n-1}} x_i^n, \tag{9}$$

[1] An alternative description of these sets proposed by the unknown referee and is given in the Appendix.

and the numbers $x_i, 1 \le i \le 2^{n-1}$, have already been determined previously for powers $1, \dots, n-1$ and the remaining numbers $x_i, 2^{n-1} < i \le 2^n$, are calculated as follows

$$x_i = y_j + k_n, \quad i = 2^{n-1} + j, \quad 1 \le i \le 2^{n-1}. \tag{10}$$

The functions $R^j(k_1, k_2, \dots, k_{n-1}), j = n-1, n$, are defined by a complete analogy:

$$R^n(k_1, k_2, \dots, k_{n-1}, k_n) = R^n(k_1, k_2, \dots, k_{n-1}) + L^n(k_1 + k_n, k_2, \dots, k_{n-1}) = \sum_{i=1}^{2^n} y_i^n, \tag{11}$$

$$R^n(k_1, k_2, \dots, k_{n-1}) = \sum_{i=1}^{2^{n-1}} y_i^n, \tag{12}$$

where $y_i, 1 \le i \le 2^{n-1}$, have already been determined and

$$y_i = x_j + k_n, \quad i = 2^{n-1} + j, \quad 1 \le i \le 2^{n-1}. \tag{13}$$

Let us write for illustration the functions $L^3(k_1, k_2, k_3)$ and $R^3(k_1, k_2, k_3)$ in their explicit form.

$$L^3(k_1, k_2, k_3) = L^3(k_1, k_2) + R^3(k_1 + k_2, k_3) =$$

$$= (a + k_1)^3 + (b + k_1)^3 + (c + k_1 + k_2)^3 + (d + k_1 + k_2)^3 +$$

$$+ (c + k_1 + k_3)^3 + (d + k_1 + k_3)^3 + (a + k_1 + k_2 + k_3)^3 + (b + k_1 + k_2 + k_3)^3,$$

$$R^3(k_1, k_2, k_3) = R^3(k_1, k_2) + L^3(k_1 + k_2, k_3) =$$

$$= (c + k_1)^3 + (d + k_1)^3 + (a + k_1 + k_2)^3 + (b + k_1 + k_2)^3 +$$

$$+ (a + k_1 + k_3)^3 + (b + k_1 + k_3)^3 + (c + k_1 + k_2 + k_3)^3 + (d + k_1 + k_2 + k_3)^3.$$

Now we are ready to formulate the first theorem.

**Theorem 1** *If $L^1(0) = R^1(0)$, i.e., (5) holds, then for all $n \ge 1$ and for all real numbers $k_1, k_2, \dots, k_n$ it follows*

$$L^n(k_1, k_2, \dots, k_n) = R^n(k_1, k_2, \dots, k_n). \tag{14}$$

**Proof.** Theorem is proved by induction. For $n = 1$ we obtain from (5), (6), and (7) that $L^1(k_1) = R^1(k_1)$.

Suppose now that (14) holds for $i$ such that $1 \le i \le n-1$ and prove that this assumption implies truth of (14) for $i = n$. To proceed we need some designations. We denote by $card\{\sum_m k_m\}$ the number of items $k_m, 1 < m \le n$, in the sum $\sum_m k_m$. We also introduce sets $S_{ij}, 1 \le i < n, 0 \le j < i$, as follows

$$S_{i0} = \{0\},$$

$$S_{ij} = \{s_{ij} : s_{ij} = \sum_m k_m, card\{\sum_m k_m\} = j, n - i < m < n\}, \quad 0 < j < i. \tag{15}$$

Thus, a set $S_{ij}$ contains as elements all possible sums $\sum_m k_m, n-i < m < n$, having $j$ items.

Now we are ready to calculate the left part of (14), i.e., $L^n(k_1,k_2,\dots,k_n)$. By using (8) and (10) we obtain

$$L^n(k_1,k_2,\dots,k_{n-1},k_n) = L^n(k_1,k_2,\dots,k_{n-1}) + \sum_{j=1}^{2^{n-1}} (y_j + k_n)^n. \tag{16}$$

By using Newton's binomial formula for every item $(y_j+k_n)^n, 1 \le j \le 2^{n-1}$, in (16) and then applying (12) and designations (15) we have

$$L^n(k_1,k_2,\dots,k_{n-1},k_n) = L^n(k_1,k_2,\dots,k_{n-1}) + R^n(k_1,k_2,\dots,k_{n-1}) +$$
$$\sum_{i=1}^{n-1} k_n^i \left(\begin{array}{c} n \\ i \end{array}\right) [\sum_{j=0}^{i-1} \sum_{s_{ij}\in S_{ij}} R^{n-i}(k_1+s_{ij},k_2,k_3,\dots,k_{n-i})] + 2^{n-1}k_n^n, \tag{17}$$

where $\left(\begin{array}{c} n \\ i \end{array}\right)$ are binomial coefficients.

We calculate $R^n(k_1,k_2,\dots,k_{n-1},k_n)$ by a complete analogy. It follows from (11) and (13) that

$$R^n(k_1,k_2,\dots,k_{n-1},k_n) = R^n(k_1,k_2,\dots,k_{n-1}) + \sum_{j=1}^{2^{n-1}} (x_j + k_n)^n. \tag{18}$$

Applying again in (18) Newton's binomial formula for items $(x_j+k_n)^n, 1 \le j \le 2^{n-1}$, and then using (9) and (15) we can write

$$R^n(k_1,k_2,\dots,k_{n-1},k_n) = R^n(k_1,k_2,\dots,k_{n-1}) + L^n(k_1,k_2,\dots,k_{n-1}) +$$
$$\sum_{i=1}^{n-1} k_n^i \left(\begin{array}{c} n \\ i \end{array}\right) [\sum_{j=0}^{i-1} \sum_{s_{ij}\in S_{ij}} L^{n-i}(k_1+s_{ij},k_2,k_3,\dots,k_{n-i})] + 2^{n-1}k_n^n, \tag{19}$$

Since numbers $n-i$ are such that $1 \le n-i < n$ then, due to our assumption, the following identities

$$L^{n-i}(k_1+s_{ij},k_2,k_3,\dots,k_{n-i}) = R^{n-i}(k_1+s_{ij},k_2,k_3,\dots,k_{n-i})$$

hold for all $0 \le j \le i-1, 1 \le i \le n-1$. This result considered together with (17) and (19) completes induction and proves the theorem. ∎

Now we are ready to obtain the main result concerning systems of the type (1) for the case $k = 2^n$.

**Theorem 2** *If $L^1(0) = R^1(0)$, i.e., (5) holds, then for all $n \ge 1$ and for all real numbers $k_1,k_2,\dots,k_n$ the numbers $x_j, y_j, 1 \le j \le 2^n$, defined by (6), (7), (10), and*

*(13) satisfy (4) and the following system*

$$\left\{\begin{array}{lcl} x_1^1+x_2^1+x_3^1+\ldots+x_{2^n-1}^1+x_{2^n}^1 & = & y_1^1+y_2^1+y_3^1+\ldots+y_{2^n-1}^1+y_{2^n}^1 \\ x_1^2+x_2^2+x_3^2+\ldots+x_{2^n-1}^2+x_{2^n}^2 & = & y_1^2+y_2^2+y_3^2+\ldots+y_{2^n-1}^2+y_{2^n}^2 \\ x_1^3+x_2^3+x_3^3+\ldots+x_{2^n-1}^3+x_{2^n}^3 & = & y_1^3+y_2^3+y_3^3+\ldots+y_{2^n-1}^3+y_{2^n}^3 \\ & \ldots\ldots & \\ x_1^n+x_2^n+x_3^n+\ldots+x_{2^n-1}^n+x_{2^n}^n & = & y_1^n+y_2^n+y_3^n+\ldots+y_{2^n-1}^n+y_{2^n}^n \end{array}\right. \tag{20}$$

**Proof.** Let us introduce auxiliary functions $M^i(k_1,k_2,\ldots,k_{n-1},k_n), 1\le i\le n$, as follows

$$M^i(k_1,k_2,\ldots,k_{n-1},k_n)=x_1^i+x_2^i+x_3^i+\ldots+x_{2^n-1}^i+x_{2^n}^i, 1\le i\le n. \tag{21}$$

From (15) and (17) we can write

$$M^i(k_1,k_2,\ldots,k_{n-1},k_n)=\sum_{j=0}^{n-i}\sum_{s_{ij}\in S_{ij}}L^i(k_1+s_{ij},k_2,k_3,\ldots,k_i).$$

Due to (14) and (11), we can rewrite this equality as

$$M^i(k_1,k_2,\ldots,k_{n-1},k_n)=\sum_{j=0}^{n-i}\sum_{s_{ij}\in S_{ij}}R^i(k_1+s_{ij},k_2,k_3,\ldots,k_i)=\sum_{j=1}^{2^n}y_j^i.$$

where $1\le i\le n$. Thus, the theorem has been completely proved. ■

Let us now establish some properties of the introduced numbers $x_m, y_m, 1\le m\le 2^n$, from (20).

**Property 1** *If there exists a number $i>1$ such that $k_i=0$ then it follows for all $n\ge i$ that for every number $x_m, 1\le m\le 2^n$, there exists such a number $y_p$ that $x_m=y_p$.*

**Proof.** Truth of the fact follows immediately from definitions (10) and (13). ■

The next property establishes presence of symmetry not only with respect to the sign '=' but also for inner subsumes of system (1).

**Property 2** *For subsumes of the numbers $x_m, y_m, 1\le m\le 2^n$, from (20) the following identities hold*

$$\sum_{j=p2^i+1}^{(p+1)2^i}x_j^i=\sum_{j=p2^i+1}^{(p+1)2^i},y_j^i, \quad 0\le p\le 2^{n-i},\ \ 1\le i\le n.$$

**Proof.** This result is a straightforward consequence of theorems 1 and 2. ■

**Property 3** *If the number* $a+b+2k_1+\sum_{i=2}^{n} k_i$ *is an integer then*

$$M^1(k_1,k_2,\ldots,k_{n-1},k_n)_{mod2} = 0.$$

**Proof.** This result is obtained from (6) – (13), since the number $M^1(k_1,k_2,\ldots,k_{n-1},k_n)$ can be written in the form

$$M^1(k_1,k_2,\ldots,k_{n-1},k_n) = \sum_{i=1}^{2^n} x_i = 2^{n-1}(a+b+2k_1+\sum_{i=2}^{n} k_i). \blacksquare$$

**Property 4** *For a fast calculation of the numbers* $M^i(k_1,k_2,\ldots,k_{n-1},k_n), 1 \le i \le n$, *from (21) the following recursive formula can be used*

$$M^i(k_1,k_2,\ldots,k_{n-1},k_n) = 2M^i(k_1,k_2,\ldots,k_{n-1})+$$

$$\sum_{i=1}^{n-1} k_n^i \binom{n}{i} M^{n-i}(k_1,k_2,\ldots,k_{n-1}) + 2^{n-1}k_n^n. \tag{22}$$

**Proof.** Property is a straightforward corollary of (19) and (21). ■

Theorem 2 has been proved for numbers $k$ from (1) equal to $2^n$. However, it is possible to generate solutions to (1) for $k \neq 2^n$ by a suitable choice of the numbers $a,b,c,d,k_1,k_2,\ldots,k_n$ in such a way that some numbers $x_j = y_i = 0$ an so can be excluded from consideration in (1). It is also possible to generate 'asymmetric' sums in sense that the number of items not equal to zero laying on the left from '=' does not coincide with that number for the sum laying right (see examples in the next section).

The following theorem extends results established above from the case of two items in the basic identity (5) to the case where $N$ items.

**Theorem 3** *If, given parameters* $k_1,k_2,\ldots,k_n$ *in the recursive process (6) - (13) the basic identity (5) is substituted by the identity*

$$\sum_{j=1}^{N} a_j = \sum_{j=1}^{N} c_j \tag{23}$$

*then, the numbers* $x_i, y_i$ *generated by the scheme will be solutions to systems (1), (2) for* $k = 2^{n-1}N$.

**Proof.** The theorem is proved by a complete analogy with the basic case $N = 2$. ■

Properties similar to the properties established for the case (5) can be proved for (23) too.

## 3 Examples and relations to the Morse-Hedlund sequence

Let us consider the first example for the case $n = 3$ with the numbers

$$a = -1, \quad b = 2.1, \quad c = 3.4, \quad d = -2.3,$$
$$k_1 = 0, \quad k_2 = 1, \quad k_3 = -0.5. \qquad (24)$$

Application of the introduced method to the data (24) gives the following piramide (4) of identities.

$$\begin{array}{rcl} (-1)^1+2.1^1 &=& 3.4^1+(-2.3)^1, \\ (-1)^2+2.1^2+4.4^2+(-1.3)^2 &=& 3.4^2+(-2.3)^2+0^2+3.1^2, \\ (-1)^3+2.1^3+4.4^3+(-1.3)^3+2.9^3+(-2.8)^3+(-0.5)^3+2.6^3 &=& 3.4^3+(-2.3)^3+0^3+3.1^3+(-1.5)^3+1.6^3+3.9^3+(-1.8)^3. \end{array}$$

The sums for the first, second and third identities are equal to 1.1, 26.46, and 111.136, respectively.

Let us now use the data from (24) and construct the corresponding systems of the type (1). This operation leads to two groups of identities presented below. The first group consists of identities for $n = 2$

$$(-1)^1+2.1^1+4.4^1+(-1.3)^1 = 3.4^1+(-2.3)^1+0^1+3.1^1,$$
$$(-1)^2+2.1^2+4.4^2+(-1.3)^2 = 3.4^2+(-2.3)^2+0^2+3.1^2$$

where sums in the first line are equal to 4.2 and in the second to 26.46. The second group of identities is written for $n = 3$.

$$(-1)^1+2.1^1+4.4^1+(-1.3)^1+2.9^1+(-2.8)^1+(-0.5)^1+2.6^1 = 3.4^1+(-2.3)^1+0^1+3.1^1+(-1.5)^1+1.6^1+3.9^1+(-1.8)^1,$$
$$(-1)^2+2.1^2+4.4^2+(-1.3)^2+2.9^2+(-2.8)^2+(-0.5)^2+2.6^2 = 3.4^2+(-2.3)^2+0^2+3.1^2+(-1.5)^2+1.6^2+3.9^2+(-1.8)^2,$$
$$(-1)^3+2.1^3+4.4^3+(-1.3)^3+2.9^3+(-2.8)^3+(-0.5)^3+2.6^3 = 3.4^3+(-2.3)^3+0^3+3.1^3+(-1.5)^3+1.6^3+3.9^3+(-1.8)^3.$$

The sums for the first, second, and third identities are equal to 6.4, 49.72, and 111.136, respectively.

The second example is in integer numbers for $n = 4$. In order to obtain repeated numbers $x_i, y_i$ and to demonstrate the nature of a 'hidden' symmetry appearance the following parameters

$$a = 1, \; b = 3, \; c = d = 2, \qquad k_1 = 0, \; k_2 = 1, \; k_3 = -2, \; k_4 = 3 \qquad (25)$$

have been chosen. The pyramid from (4) for these data is shown below.

$$\begin{array}{rcl} 1^1+3^1 &=& 2^1+2^1, \\ 1^2+3^2+3^2+3^2 &=& 2^2+2^2+2^2+4^2, \\ 1^3+3^3+3^3+3^3+0^3+0^3+0^3+2^3 &=& 2^3+2^3+2^3+4^3+(-1)^3+1^3+1^3+1^3, \\ 1^4+3^4+3^4+3^4+0^4+0^4+0^4+2^4+5^4+5^4+5^4+7^4+2^4+4^4+4^4+4^4 &=& 2^4+2^4+2^4+4^4+(-1)^4+1^4+1^4+1^4+4^4+6^4+6^4+6^4+3^4+3^4+3^4+5^4. \end{array}$$

The sums for the first, second, third, and forth identities are equal, respectively, to 4, 28, 90, and 5320. In this record all the numbers $x_i, y_i$ have been shown. Let us now eliminate from the record items equal to zero and elements $x_i$ and $y_j$ such that $x_i = y_j$. In this case the symmetry is present only in an implicit form and is not seen from the record:

$$\begin{array}{rcl} 1^1+3^1 &=& 2^1+2^1, \\ 1^2+3^2+3^2+3^2 &=& 2^2+2^2+2^2+4^2, \\ 3^3+3^3+3^3 &=& 2^3+2^3+4^3+(-1)^3+1^3+1^3, \\ 5^4+5^4+7^4+4^4 &=& 2^4+(-1)^4+1^4+1^4+6^4+6^4 \end{array}$$

Let us consider now how does system (20) look like for data (25).

$$\begin{array}{rcl} 1^1{+}3^1{+}3^1{+}3^1{+}0^1{+}0^1{+}0^1{+}2^1{+}5^1{+}5^1{+}5^1{+}7^1{+}2^1{+}4^1{+}4^1{+}4^1 &=& 2^1{+}2^1{+}2^1{+}4^1{+}(-1)^1{+}1^1{+}1^1{+}1^1{+}4^1{+}6^1{+}6^1{+}6^1{+}3^1{+}3^1{+}3^1{+}5^1, \\ 1^2{+}3^2{+}3^2{+}3^2{+}0^2{+}0^2{+}0^2{+}2^2{+}5^2{+}5^2{+}5^2{+}7^2{+}2^2{+}4^2{+}4^2{+}4^2 &=& 2^2{+}2^2{+}2^2{+}4^2{+}(-1)^2{+}1^2{+}1^2{+}1^2{+}4^2{+}6^2{+}6^2{+}6^2{+}3^2{+}3^2{+}3^2{+}5^2, \\ 1^3{+}3^3{+}3^3{+}3^3{+}0^3{+}0^3{+}0^3{+}2^3{+}5^3{+}5^3{+}5^3{+}7^3{+}2^3{+}4^3{+}4^3{+}4^3 &=& 2^3{+}2^3{+}2^3{+}4^3{+}(-1)^3{+}1^3{+}1^3{+}1^3{+}4^3{+}6^3{+}6^3{+}6^3{+}3^3{+}3^3{+}3^3{+}5^3, \\ 1^4{+}3^4{+}3^4{+}3^4{+}0^4{+}0^4{+}0^4{+}2^4{+}5^4{+}5^4{+}5^4{+}7^4{+}2^4{+}4^4{+}4^4{+}4^4 &=& 2^4{+}2^4{+}2^4{+}4^4{+}(-1)^4{+}1^4{+}1^4{+}1^4{+}4^4{+}6^4{+}6^4{+}6^4{+}3^4{+}3^4{+}3^4{+}5^4. \end{array}$$

The sums are equal to 48, 208, 1008, and 5320. Deleting the explicit symmetry we obtain the following identities where the symmetry is hidden.

$$\begin{array}{rcl} 5^1+5^1+7^1+4^1 &=& 2^1+(-1)^1+1^1+1^1+6^1+6^1, \\ 5^2+5^2+7^2+4^2 &=& 2^2+(-1)^2+1^2+1^2+6^2+6^2, \\ 5^3+5^3+7^3+4^3 &=& 2^3+(-1)^3+1^3+1^3+6^3+6^3, \\ 5^4+5^4+7^4+4^4 &=& 2^4+(-1)^4+1^4+1^4+6^4+6^4 \end{array}$$

The corresponding sums are equal to 21, 115, 657, and 3907.

Let us now illustrate Theorem 3 by an example having $N = 3$ in (23) and shown in Table 1 for the power $n = 4$. In this example the following parameters have been used:

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = 7, \qquad\qquad c_1 = 2, \quad c_2 = 4, \quad c_3 = 5,$$

$$k_1 = 0, \quad k_2 = -1, \quad k_3 = 1.3, \quad k_4 = -2.5.$$

The last line of Table 1 presents the values

$$\sum_{i=1}^{24} x_i^j, \qquad \sum_{i=1}^{24} y_i^j, \qquad 1 \leq j \leq 4,$$

showing that the numbers $x_i, y_i, 1 \leq j \leq 4$, are solutions to system (1) for $n = 4$ and $k = 24$.

Table 1 illustrates also Property 2. In fact, it is split in boxes that contain groups of elements of the sequences $\{x_i^j\}$ and $\{y_i^j\}, 1 \leq j \leq 4$, having equal sums. For example, for the power 2, the box containing the numbers $x_7^2, x_8^2, \ldots, x_{11}^2, x_{12}^2$ and $y_7^2, y_2^8, \ldots, y_{11}^2, y_{12}^2$ shows that

$$x_7^2 + x_8^2 + \ldots + x_{11}^2 + x_{12}^2 = y_7^2 + y_8^2 + \ldots + y_{11}^2 + y_{12}^2.$$

Table 1: Example for $n = 4$ and the starting identity $1+3+7 = 2+4+5$

| | $x^1$ | $y^1$ | $x^2$ | $y^2$ | $x^3$ | $y^3$ | $x^4$ | $y^4$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 4 | 1 | 8 | 1 | 16 |
| 2 | 3 | 4 | 9 | 16 | 27 | 64 | 81 | 256 |
| 3 | 7 | 5 | 49 | 25 | 343 | 125 | 2401 | 625 |
| 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 5 | 3 | 2 | 9 | 4 | 27 | 8 | 81 | 16 |
| 6 | 4 | 6 | 16 | 36 | 64 | 216 | 256 | 1296 |
| 7 | 3,3 | 2,3 | 10,89 | 5,29 | 35,937 | 12,167 | 118,5921 | 27,9841 |
| 8 | 5,3 | 4,3 | 28,09 | 18,49 | 148,877 | 79,507 | 789,0481 | 341,8801 |
| 9 | 6,3 | 8,3 | 39,69 | 68,89 | 250,047 | 571,787 | 1575,2961 | 4745,8321 |
| 10 | 1,3 | 2,3 | 1,69 | 5,29 | 2,197 | 12,167 | 2,8561 | 27,9841 |
| 11 | 3,3 | 4,3 | 10,89 | 18,49 | 35,937 | 79,507 | 118,5921 | 341,8801 |
| 12 | 7,3 | 5,3 | 53,29 | 28,09 | 389,017 | 148,877 | 2839,8241 | 789,0481 |
| 13 | -0,5 | -1,5 | 0,25 | 2,25 | -0,125 | -3,375 | 0,0625 | 5,0625 |
| 14 | 1,5 | 0,5 | 2,25 | 0,25 | 3,375 | 0,125 | 5,0625 | 0,0625 |
| 15 | 2,5 | 4,5 | 6,25 | 20,25 | 15,625 | 91,125 | 39,0625 | 410,0625 |
| 16 | -2,5 | -1,5 | 6,25 | 2,25 | -15,625 | -3,375 | 39,0625 | 5,0625 |
| 17 | -0,5 | 0,5 | 0,25 | 0,25 | -0,125 | 0,125 | 0,0625 | 0,0625 |
| 18 | 3,5 | 1,5 | 12,25 | 2,25 | 42,875 | 3,375 | 150,0625 | 5,0625 |
| 19 | -0,2 | 0,8 | 0,04 | 0,64 | -0,008 | 0,512 | 0,0016 | 0,4096 |
| 20 | 1,8 | 2,8 | 3,24 | 7,84 | 5,832 | 21,952 | 10,4976 | 61,4656 |
| 21 | 5,8 | 3,8 | 33,64 | 14,44 | 195,112 | 54,872 | 1131,6496 | 208,5136 |
| 22 | -0,2 | -1,2 | 0,04 | 1,44 | -0,008 | -1,728 | 0,0016 | 2,0736 |
| 23 | 1,8 | 0,8 | 3,24 | 0,64 | 5,832 | 0,512 | 10,4976 | 0,4096 |
| 24 | 2,8 | 4,8 | 7,84 | 23,04 | 21,952 | 110,592 | 61,4656 | 530,8416 |
| Sum | 61,6 | 61,6 | 305,08 | 305,08 | 1599,724 | 1599,724 | 9712,6972 | 9712,6972 |

To conclude this example let us note, that $x_6 = y_2 = 4$. Elimination of this couple of numbers from consideration gives us a solution to (1) for $n = 4$ and $k = 23$. We can repeat this procedure for pairs

$$x_8 = y_{12} = 5.3, \qquad x_{14} = y_{18} = 1.5, \qquad x_{24} = y_{20} = 2.8$$

arriving to a solution to (1) for $n = 4$ and $k = 20$.

The introduced method for generation of symmetric exponential sums has interesting relations to the Morse-Hedlund sequences [19] (being a particular case of the Prouhet sequences [20]) and to the theory of magic cubes developed in [1, 2]. Let us write the first items of the Morse-Hedlund sequence (see Table 2) and numerate its items by natural numbers.

It has been noticed (see, for example, [14, 20]) that this sequence splits the natural numbers staying in the second line of Table 2 in two groups satisfying (1).

Table 2: The Morse-Hedlund sequence

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | ... |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | ... |

Table 3: The half-completed magic square based on the Morse-Hedlund sequence

| | | | |
|---|---|---|---|
| | 2 | 3 | |
| 5 | | | 8 |
| 9 | | | 12 |
| | 14 | 15 | |

The first group contains the numbers corresponding to '1' in the Morse-Hedlund sequence and the second group contains the numbers corresponding to '0'. Thus, for the first 16 numbers we obtain

$$\begin{aligned} 1^1+4^1+6^1+7^1+10^1+11^1+13^1+16^1 &= 2^1+3^1+5^1+8^1+9^1+12^1+14^1+15^1, \\ 1^2+4^2+6^2+7^2+10^2+11^2+13^2+16^2 &= 2^2+3^2+5^2+8^2+9^2+12^2+14^2+15^2, \\ 1^3+4^3+6^3+7^3+10^3+11^3+13^3+16^3 &= 2^3+3^3+5^3+8^3+9^3+12^3+14^3+15^3. \end{aligned}$$

It can be seen immediately that these identities are a particular case of our results with

$$a=1, \quad b=4, \quad c=2, \quad d=3, \qquad k_1=0, \quad k_2=2^2, \quad k_3=2^3. \tag{26}$$

Naturally, by taking $k_n=2^n$ it is possible to generate Prouhet identities for any $n$.

Let us now establish relations of the introduced method to the theory of magic squares (and cubes) developed in [1, 2]. A $T\times T$ matrix is called a *magic square of order $T$* if the sum of the entries in any row, column, or diagonal is the same number. Usually, the entries of the magic square are taken from the numbers $1,2,\ldots,T^2$ and each of the numbers is used exactly once as an entry.

There exist different ways to construct magic squares (see [3]). Let us consider the one proposed in [1, 2] and based on the Morse-Hedlund sequence. In this paper we consider only the core case $T=4$ being a basis for construction of the squares (and cubes) of the order $T=2^n$ (see [1, 2] for details).

The algorithm to make the $4\times 4$ magic square is the following. Count through the boxes in the square, and whenever the number you count corresponds to '0' in

Table 4: The completed magic square based on the Morse-Hedlund sequence

| | | | |
|---|---|---|---|
| 16 | 2 | 3 | 13 |
| 5 | 11 | 10 | 8 |
| 9 | 7 | 6 | 12 |
| 4 | 14 | 15 | 1 |

Table 5: This matrix is a magic square for all real numbers $a,b,c$, and $d$ satisfying (5) and any real numbers $k_1$ and $k_2$

| | | | |
|---|---|---|---|
| $d+k_1+k_2$ | $a$ | $b$ | $c+k_1+k_2$ |
| $c+k_1$ | $b+k_2$ | $a+k_2$ | $d+k_1$ |
| $c+k_2$ | $b+k_1$ | $a+k_1$ | $d+k_2$ |
| $d$ | $a+k_1+k_2$ | $b+k_1+k_2$ | $c$ |

Table 6: The magic square with irrational entries obtained by using the data (27)

| | | | |
|---|---|---|---|
| $5+\sqrt{2}+\sqrt{3}$ | $0$ | $6$ | $1+\sqrt{2}+\sqrt{3}$ |
| $1+\sqrt{2}$ | $6+\sqrt{3}$ | $\sqrt{3}$ | $5+\sqrt{2}$ |
| $1+\sqrt{3}$ | $6+\sqrt{2}$ | $\sqrt{2}$ | $5+\sqrt{3}$ |
| $5$ | $\sqrt{2}+\sqrt{3}$ | $6+\sqrt{2}+\sqrt{3}$ | $1$ |

the Morse-Hedlund sequence (see Table 2), write the number down in the box. The rest of the boxes is filled in by counting numbers from 16 to 1 and gives the magic square presented in Table 4.

Since the Prouhet identities were obtained as a particular case of our method with the data from (26) used in construction of the magic square in Table 4, let us execute the inverse operation and substitute in Table 4 the numbers $1, 2, \ldots, 16$ by the corresponding explicit formulae for the sets $\{x_i : 1 \leq i \leq 8\}$ and $\{y_i : 1 \leq i \leq 8\}$. The result is shown in Table 5 were, of course, the numbers $a, b, c$, and $d$ satisfy (5).

It can be shown that, due to (5), the sum of the entries in any row, column, or diagonal is equal to $2(a+b+k_1+k_2)$. Thus, the obtained matrix is a magic square for all real numbers $a, b, c$, and $d$ satisfying (5) and any real numbers $k_1$ and $k_2$. Of course, some combination of these parameters can give repeating numbers. Note that the data from (26) give us the the magic square from Table 4 as a particular case of Table 5.

We conclude the paper by an example with irrational numbers.

$$a=0, \quad b=6, \quad c=1, \quad d=5, \qquad k_1=\sqrt{2}, \quad k_2=\sqrt{3}. \tag{27}$$

For these data the corresponding systems (1) and (4) become

$$\begin{aligned}
1^1+5^1+\sqrt{2}^1+(6+\sqrt{2})^1+\sqrt{3}^1+(6+\sqrt{3})^1+(1+\sqrt{2}+\sqrt{3})^1+(5+\sqrt{2}+\sqrt{3})^1 &= 0^1+6^1+(1+\sqrt{2})^1+(5+\sqrt{2})^1+(1+\sqrt{3})^1+(5+\sqrt{3})^1+(\sqrt{2}+\sqrt{3})^1+(6+\sqrt{2}+\sqrt{3})^1, \\
1^2+5^2+\sqrt{2}^2+(6+\sqrt{2})^2+\sqrt{3}^2+(6+\sqrt{3})^2+(1+\sqrt{2}+\sqrt{3})^2+(5+\sqrt{2}+\sqrt{3})^2 &= 0^2+6^2+(1+\sqrt{2})^2+(5+\sqrt{2})^2+(1+\sqrt{3})^2+(5+\sqrt{3})^2+(\sqrt{2}+\sqrt{3})^2+(6+\sqrt{2}+\sqrt{3})^2, \\
1^3+5^3+\sqrt{2}^3+(6+\sqrt{2})^3+\sqrt{3}^3+(6+\sqrt{3})^3+(1+\sqrt{2}+\sqrt{3})^3+(5+\sqrt{2}+\sqrt{3})^3 &= 0^3+6^3+(1+\sqrt{2})^3+(5+\sqrt{2})^3+(1+\sqrt{3})^3+(5+\sqrt{3})^3+(\sqrt{2}+\sqrt{3})^3+(6+\sqrt{2}+\sqrt{3})^3
\end{aligned}$$

and

$$\begin{aligned}
1^1+5^1 &= 0^1+6^1, \\
1^2+5^2+\sqrt{2}^2+(6+\sqrt{2})^2 &= 0^2+6^2+(1+\sqrt{2})^2+(5+\sqrt{2})^2, \\
1^3+5^3+\sqrt{2}^3+(6+\sqrt{2})^3+\sqrt{3}^3+(6+\sqrt{3})^3+(1+\sqrt{2}+\sqrt{3})^3+(5+\sqrt{2}+\sqrt{3})^3 &= 0^3+6^3+(1+\sqrt{2})^3+(5+\sqrt{2})^3+(1+\sqrt{3})^3+(5+\sqrt{3})^3+(\sqrt{2}+\sqrt{3})^3+(6+\sqrt{2}+\sqrt{3})^3.
\end{aligned}$$

Finally, the magic square obtained from the data (27) by using the general method from Table 5 is shown in Table 6.

# 4 Appendix

This Appendix contains an alternative description of the sets $\{x_i : 1 \le i \le k\}$ and $\{y_i : 1 \le i \le k\}$ proposed by the unknown referee.

Given $n$, we construct $x_i, 1 \le i \le 2^{n+1}$, and $y_i, 1 \le i \le 2^{n+1}$, so that for $1 \le m \le n+1$,

$$\sum_{i=1}^{2^{n+1}} x_i^m = \sum_{i=1}^{2^{n+1}} y_i^m \tag{28}$$

Choose $a, b, c$, and $d$ so that $a+b = c+d$. Choose any $n$ values $k_1, \ldots, k_n$. The $x_i$ are obtained by adding $a$ to an even subset of the $k_j$ (including the empty set) or $b$ to an even subset of the $k_j$ or $c$ to an odd subset of the $k_j$ or $d$ to an odd subset of the $k_j$. The $y_i$ are obtained by adding either $a$ or $b$ to an odd subset of the $k_j$ and $c$ or $d$ to an even subset of the $k_j$.

Equation (28) is then equivalent to the statement that the function of $a$ and $b$ given by

$$f(a,b) = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|} \left[ \left( a + \sum_{i \in S} k_i \right)^m + \left( b + \sum_{i \in S} k_i \right)^m \right], \tag{29}$$

is a function of $a+b$ when $1 \le m \le n+1$.

A simple expansion shows that

$$\begin{aligned}
f(a,b) &= (a^m + b^m) \sum_S (-1)^{|S|} + \sum_{S \neq \emptyset} (-1)^{|S|} \sum_{j=0}^{m-1} (a^j + b^j) \binom{m}{j} \left( \sum_{i \in S} k_i \right)^{m-j} \\
&= \sum_{j=0}^{m-1} (a^j + b^j) \binom{m}{j} \sum_{S \neq \emptyset} (-1)^{|S|} \left( \sum_{i \in S} k_i \right)^{m-j} \\
&= \sum_{j=0}^{m-1} (a^j + b^j) \binom{m}{j} \sum_{S = \{s_1,\ldots,s_{|S|}\} \neq \emptyset} (-1)^{|S|} \sum_{i_1+\ldots+i_{|S|} = m-j} \binom{m-j}{i_1,\ldots,i_{|S|}} \prod_{j=1}^{|S|} k_{s_j}^{i_j} \\
&= \sum_{j=0}^{m-1} (a^j + b^j) \binom{m}{j} \sum_{i_1+\ldots+i_n = m-j} \binom{m-j}{i_1,\ldots,i_n} \prod_{j=1}^{n} x_j^{i_j} \sum_{S \ni i_j > 0 \Rightarrow j \in S} (-1)^{|S|} \\
&= \sum_{j=0}^{m-1} (a^j + b^j) \binom{m}{j} \sum_{\substack{i_1+\ldots+i_n = m-j \\ \forall j,\ i_j \ge 1}} \binom{m-j}{i_1,\ldots,i_n} \prod_{j=1}^{n} k_j^{i_j} (-1)^n \\
&= \begin{cases} (-1)^n (n+1)!\ x_1 \cdots x_n (a+b+\sum_{i=1}^n k_i), & \text{if } m = n+1, \\ (-1)^n (n)!\ k_1 \cdots k_n, & \text{if } m = n, \\ 0, & \text{if } m < n. \end{cases}
\end{aligned}$$

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## References


[1] A. Adler (1992) Magic cubes and 3-Adic Zeta function, The Mathematical Intelligencer, 14(3), 14–23.

[2] A. Adler and S.-Y.R. Li (1977) Magic cubes and Prouhet sequences, American Mathematical Monthly, 84, 618–627.

[3] W.S. Andrews (1960) Magic squares and cubes, Dover, New York.

[4] G.I. Arkhipov, A.A. Karatsuba and V.N. Chubarikov (1987) Theory of multiple trigonometric sums, Nauka, Moskow.

[5] M. Boldrin and M. Woodford (1990) Equlibrium models displaying endogenous fluctuations and chaos: A survey, Journal of Monetary Economics, 25, 189–220.

[6] G.J. Chaitin (1966) On the length of programs for computing finite binary sequences, J. Assoc. Comp. Math., 13, 547–570.

[7] M.J. Feigenbaum (1980) Universal behavior in nonlinear systems, Los Alamos Science 1(1), 4–27.

[8] D. Hilbert (1909) Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen, Math. Ann., 67, 281–300.

[9] R. Horst and P.M. Pardalos (1995) Handbook of Global Optimization, Kluwer Academic Publishers, Dordrecht.

[10] E. Kamke (1921) Verallgeme inerungen des Waring - Hilbertschen Satzes, Math. Ann., 3, 3–38.

[11] A.A. Karatsuba (1987) Hilbert-Kamke's problem in the analitical number theory, Math. Zametki, 41(2), 272–284.

[12] A.N. Kolmogorov (1965) Three approaches to the quantitative definition of information, Problems of Inform. Transmission, 1, 3–7.

[13] N.M. Korobov (1989) Trigonometric sums and their applications, Nauka, Moskow.

[14] V.V. Korotkich (1993) An approach to nonlinear systems cycles structural properties description, Institute for Automated Systems, Moscow.

[15] V.V. Korotkich (1999) A Mathematical Structure for Emergent Computation, Kluwer Academic Publishers, Dordrecht.

[16] H.J. Korsch and H.-J. Jodl (1994) Chaos, Springer-Verlag, Berlin.

[17] D.H. Lehmer (1947) The Tarry-Escott problem, Scripta Math., 14, 37–41.

[18] L. Mordell (1932) On a sum analogous to a Gauss's sum, Quart. J. Math., 3, 161–167.

[19] M. Morse and G.A. Hedlund (1944) Unending chess, symbolic dynamics and a problem in semigroups, Duke Math. J., 11, 1–7.

[20] E. Prouhet (1851) Memoire sur quelques relations entre les puissances des nombres, C.R. Acad. Sci., Paris, 33, 225.

[21] R.J. Solomonoff (1964) A formal theory of inductive inference, Information and Control 7(1), 1–22.

[22] R. G. Strongin and Ya. D. Sergeyev (2000) Global Optimization with Non-Convex Constraints: Sequential and Parallel Algorithms, Kluwer Academic Publishers, Dordrecht.

[23] R.C. Vaughan (1979) A survey of some important problems in additive number theory, Societe Math. de France, Asterisque 61, 213–222.

[24] E. Waring (1770) Meditationes algebraicae, Cambridge.