Computations with grossone-based infinities

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Abstract. In this paper, a recent computational methodology is described. It has been introduced with the intention to allow one to work with infinities and infinitesimals numerically in a unique computational framework. It is based on the principle 'The part is less than the whole' applied to all quantities (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite). The methodology uses as a computational device the Infinity Computer (patented in USA and EU) working numerically with infinite and infinitesimal numbers that can be written in a positional system with an infinite radix. On a number of examples dealing mainly with infinite sets and Turing machines with different infinite tapes it is shown that it becomes possible to execute a fine analysis of these mathematical objects. The accuracy of the obtained results is continuously compared with results obtained by traditional tools used to work with mathematical objects involving infinity.

Keywords: Numbers and numerals, numerical infinities and infinitesimals, infinite sets, Turing machines, infinite sequences.

1 Introduction

There exists an important distinction between *numbers* and *numerals*. A *numeral* is a symbol (or a group of symbols) that represents a *number*. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '10', 'ten', 'IIIIIIIIII','X', '=', and 'Ĩ' are different numerals, but they all represent the same number[1]. Rules

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[1] The last two numerals, = and Ĩ, are probably less known. The former belongs to the Maya numeral system where one horizontal line indicates five and two lines one above the other indicate ten. Dots are added above the lines to represent additional units. For instance, $\doteq$ means eleven in this numeral system. The latter symbol, Ĩ, belongs to the Cyrillic numeral system derived from the Cyrillic script. This numeral system was developed in the late X^{th} century and was used by South and East Slavic peoples. The system was used in Russia as late as the early $XVIII^{th}$ century when

used to write down numerals together with algorithms for executing arithmetical operations form a *numeral system*.

In our everyday activities with finite numbers the *same* finite numerals are used for *different* purposes (e.g., the same numeral 10 can be used to express the number of elements of a set, to indicate the position of an element in a sequence, and to execute practical computations). In contrast, when we face the necessity to work with infinities or infinitesimals, the situation changes drastically. In fact, in this case *different* numerals are used to work with infinities and infinitesimals in *different* situations. To illustrate this fact it is sufficient to mention that we use the symbol ∞ in standard analysis, ω for working with ordinals, $\aleph_0, \aleph_1, ...$ for dealing with cardinalities.

Many theories dealing with infinite and infinitesimal quantities have a symbolic (not numerical) character. For instance, many versions of non-standard analysis (see [23]) are symbolic, since they have no numeral systems to express their numbers by a finite number of symbols (the finiteness of the number of symbols is necessary for organizing numerical computations). Namely, if we consider a finite n than it can be taken $n = 134$, or $n = 65$ or any other numeral used to express finite quantities and consisting of a finite number of symbols. In contrast, if we consider a non-standard infinite m then it is not clear which numerals can be used to assign a concrete value to m.

Analogously, in non-standard analysis, if we consider an infinitesimal h then it is not clear which numerals consisting of a finite number of symbols can be used to assign a value to h and to write $h = ...$ In fact, very often in non-standard analysis texts, a *generic* infinitesimal h is used and it is considered as a symbol, i.e., only symbolic computations can be done with it. Approaches of this kind leave unclear such issues, e.g., whether the infinite $1/h$ is integer or not or whether $1/h$ is the number of elements of an infinite set. If one wishes to consider two infinitesimals h_1 and h_2 then it is not clear how to compare them because numeral systems that can express infinitesimals are not provided by non-standard analysis techniques. In fact, when we work with finite quantities, then we can compare x and y if they assume numerical values, e.g., $x = 25$ and $y = 78$ then, by using rules of the numeral system the symbols 25 and 78 belong to, we can compute that $y > x$.

Even though there exist codes allowing one to work symbolically with ∞ and other symbols related to the concepts of infinity and infinitesimals, traditional computers work numerically only with finite numbers and situations where the usage of infinite or infinitesimal quantities is required are studied mainly theoretically (see [2, 3, 8, 10, 11, 15, 16, 23, 45] and references given therein). The fact that numerical computations with infinities and infinitesimals have not been implemented so far on computers can be explained by several difficulties. Obviously, among them we can mention the fact that arithmetics developed for this purpose are quite different with respect to the way of computing we use when

it was replaced with Arabic numerals. To distinguish numbers from text, a titlo, ˜ , is drawn over the symbols showing so that this is a numeral and, therefore, it represents a number and not just a character of text.

we deal with finite quantities. For instance, there exist undetermined operations ($\infty - \infty$, $\frac{\infty}{\infty}$, etc.) that are absent when we work with finite numbers. There exist also practical difficulties that preclude an implementation of numerical computations with infinity and infinitesimals. For example, it is not clear how to store an infinite quantity in a finite computer memory.

A computational methodology introduced recently in [26, 32, 36, 40] allows one to look at infinities and infinitesimals in a new way and to execute *numerical* computations with infinities and infinitesimals on the Infinity Computer patented in USA (see [30]) and other countries. Moreover, this approach proposes a numeral system that uses *the same numerals* for several different purposes for dealing with infinities and infinitesimals: for measuring infinite sets; for indicating positions of elements in ordered infinite sequences; for working with functions and their derivatives that can assume different infinite, finite, and infinitesimal values and can be defined over infinite and infinitesimal domains; for describing Turing machines, etc.

An international scientific community developing a number of interesting theoretical and practical applications in several research areas by using the new methodology grows rapidly. Among these studies it is worthy to mention papers connecting the new approach to the historical panorama of ideas dealing with infinities and infinitesimals (see [17–19, 41]). In particular, relations of the new approach to bijections are studied in [19] and metamathematical investigations on the new theory and its non-contradictory can be found in [18]. Then, the new methodology has been applied for studying Euclidean and hyperbolic geometry (see [20, 21]), percolation (see [12, 13, 44]), fractals (see [25, 27, 35, 44]), numerical differentiation and optimization (see [4, 28, 33, 47]), infinite series and the Riemann zeta function (see [29, 34, 46]), the first Hilbert problem, Turing machines, and lexicographic ordering (see [31, 41–43, 39]), cellular automata (see [5–7]), ordinary differential equations (see [37, 38]), etc. The interested reader is invited to have a look also at surveys [26, 32, 36] and the book [24] written in a popular way.

In this paper, we briefly describe the new methodology and the numeral system showing how they can be used in a number of situations where infinities and infinitesimals are useful. Infinite sets, bijections, and Turing machines are mainly discussed.

2 Numeral systems, their accuracy, and numbers they can express

It is necessary to remind that different numeral systems can express different sets of numbers and they can be more or less suitable for executing arithmetical operations. Even the powerful positional system is not able to express, e.g., the number $\sqrt{2}$ by a finite number of symbols (the finiteness is essential for executing numerical computations) and this special numeral, $\sqrt{2}$, is deliberately introduced to express the desired quantity. There exist many numeral systems that are weaker than the positional one. For instance, Roman numeral system is not able to express zero and negative numbers and such expressions as III − VIII or X-X are indeterminate forms in this numeral system. As a result, before appearing the positional numeral system and inventing zero mathematicians were not able to create theorems involving zero and negative numbers and to execute computations with them. Thus, numeral systems seriously bound the possibilities of human beings to compute and developing new, more powerful than existing ones, numeral systems can help a lot both in theory and practice of computations.

Even though Roman numeral system is weaker than the positional one it is not the weakest numeral system. There exist really feeble numeral systems allowing their users to express very few numbers and one of them is illuminating for our study. This numeral system is used by a tribe, Pirahã, living in Amazonia nowadays. A study published in *Science* in 2004 (see [9]) describes that these people use an extremely simple numeral system for counting: one, two, many. For Pirahã, all quantities larger than two are just 'many' and such operations as 2+2 and 2+1 give the same result, i.e., 'many'. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, and 5, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that.

It is worthy to mention that the result 'many' is not wrong. It is just *inaccurate*. Analogously, when we observe a garden with 343 trees, then both phrases: 'There are 343 trees in the garden' and 'There are many trees in the garden' are correct. However, the accuracy of the former phrase is higher than the accuracy of the latter one. Thus, the introduction of a numeral system having numerals for expressing numbers 3 and 4 leads to a higher accuracy of computations and allows one to distinguish results of operations 2+1 and 2+2.

The poverty of the numeral system of Pirahã leads also to the following results

$$\text{`many'} + 1 = \text{`many'}, \qquad \text{`many'} + 2 = \text{`many'},$$

$$\text{`many'} - 1 = \text{`many'}, \qquad \text{`many'} - 2 = \text{`many'},$$

$$\text{`many'} + \text{`many'} = \text{`many'}$$

that are crucial for changing our outlook on infinity. In fact, by changing in these relations 'many' with ∞ we get relations used to work with infinity in the traditional calculus and Cantor's cardinals

$$\infty + 1 = \infty, \quad \infty + 2 = \infty, \quad \infty - 1 = \infty, \quad \infty - 2 = \infty, \quad \infty + \infty = \infty,$$

$$\aleph_0 + 1 = \aleph_0, \quad \aleph_0 + 2 = \aleph_0, \quad \aleph_0 - 1 = \aleph_0, \quad \aleph_0 - 2 = \aleph_0, \quad \aleph_0 + \aleph_0 = \aleph_0.$$

It should be mentioned that the astonishing numeral system of Pirahã is not an isolated example of this way of counting. In fact, the same counting system, one, two, many, is used by the Warlpiri people, aborigines living in the Northern Territory of Australia (see [1]). The Pitjantjatjara people living in the Central Australian desert use numerals one, two, three, big mob (see [14]) where 'big mob' works as 'many'. It makes sense to remind also another Amazonian tribe

– Mundurukú (see [22]) who fail in exact arithmetic with numbers larger than 5 but are able to compare and add large approximate numbers that are far beyond their naming range. Particularly, they use the words 'some, not many' and 'many, really many' to distinguish two types of large numbers. Their arithmetic with 'some, not many' and 'many, really many' reminds strongly the rules Cantor uses to work with $\aleph_0$ and $\aleph_1$, respectively. For instance, compare

'some, not many'+ 'many, really many' = 'many, really many'

with

$$\aleph_0 + \aleph_1 = \aleph_1.$$

This comparison suggests that our difficulty in working with infinity is not connected to the *nature* of infinity but is a result of inadequate numeral systems used to express infinite numbers. Traditional numeral systems have been developed to express finite quantities and they simply have no sufficiently high number of numerals to express different infinities (and infinitesimals). In other words, the difficulty we face is not connected to the object of our study – infinity – but is the result of weak instruments – numeral systems – used for our study.

The way of reasoning where the object of the study is separated from the tool used by the investigator is very common in natural sciences where researchers use tools to describe the object of their study and the used instrument influences the results of the observations and determine their accuracy. When a physicist uses a weak lens A and sees two black dots in his/her microscope he/she does not say: The object of the observation *is* two black dots. The physicist is obliged to say: the lens used in the microscope allows us to see two black dots and it is not possible to say anything more about the nature of the object of the observation until we change the instrument - the lens or the microscope itself - by a more precise one. Suppose that he/she changes the lens and uses a stronger lens B and is able to observe that the object of the observation is viewed as eleven (smaller) black dots. Thus, we have two different answers: (i) the object is viewed as two dots if the lens A is used; (ii) the object is viewed as eleven dots by applying the lens B. Both answers are correct but with the *different accuracies* that depend on the lens used for the observation.

The same happens in Mathematics studying natural phenomena, numbers, objects that can be constructed by using numbers, sets, etc. Numeral systems used to express numbers are among the instruments of observations used by mathematicians. As we have illustrated above, the usage of powerful numeral systems gives the possibility to obtain more precise results in Mathematics in the same way as usage of a good microscope gives the possibility of obtaining more precise results in Physics.

3 Grossone-based numerals

In order to increase the accuracy of computations with infinities and infinitesimals, the computational methodology developed in [24, 26, 32] proposes a numeral system that allows one to observe infinities and infinitesimals with a higher accuracy. This numeral system avoids situations similar to 'many' + 1 = 'many' and $\infty - 1 = \infty$ providing results ensuring that if a is a numeral written in this numeral system then for any a (i.e., a can be finite, infinite, or infinitesimal) it follows $a + 1 > a$ and $a - 1 < a$.

The numeral system is based on a new infinite unit of measure expressed by the numeral ① called *grossone* that is introduced as the number of elements of the set of natural[2] numbers

$$\mathbb{N} = \{1, 2, 3, \dots \}. \quad (1)$$

Concurrently with the introduction of ① in the mathematical language all other symbols (like ∞, Cantor's ω, $\aleph_0, \aleph_1, ...$, etc.) traditionally used to deal with infinities and infinitesimals are excluded from the language because ① and other numbers constructed with its help not only can be used instead of all of them but can be used with a higher accuracy. Analogously, when zero and the positional numeral system had been introduced in Europe, Roman numerals I, V, X, etc. had not been involved and new symbols 0, 1, 2, etc. have been used to express numbers. The new element – zero expressed by the numeral 0 – had been introduced by describing its properties in the form of axioms. Analogously, ① is introduced by describing its properties postulated by the Infinite Unit Axiom added to axioms for real numbers (see [26, 32] for a detailed discussion). Let us comment upon some of properties of ①.

If we consider a finite integer k, then the number of elements of the set $\{1, 2, 3, \dots k-1, k\}$ is its largest element, i.e., k. For instance, the number 4 in the set

$$A = \{1, 2, 3, 4\} \quad (2)$$

is the largest element in the set A and the number of elements of A. Grossone has been introduced as the number of elements of the set of natural numbers and, therefore, we have the same situation as in (2), i.e., $① \in \mathbb{N}$. As a consequence, the introduction of ① allows us to write down the set of natural numbers as follows

$$\mathbb{N} = \{1, 2, \ \dots \ \frac{①}{2} - 2, \frac{①}{2} - 1, \frac{①}{2}, \frac{①}{2} + 1, \frac{①}{2} + 2, \ \dots \ ① - 2, \ ① - 1, \ ①\}. \quad (3)$$

Infinite natural numbers

$$\dots \ \frac{①}{2} - 2, \frac{①}{2} - 1, \frac{①}{2}, \frac{①}{2} + 1, \frac{①}{2} + 2, \ \dots \ ① - 2, ① - 1, ① \quad (4)$$

that are invisible if traditional numeral systems are used to observe the set of natural numbers can be viewed now thanks to the introduction of ①. The two records, (1) and (3), refer to the same set – the set of natural numbers – and

[2] Notice that nowadays not only positive integers but also zero is frequently included in $\mathbb{N}$. However, since zero has been invented significantly later than positive integers used for counting objects, zero is not include in $\mathbb{N}$ in this text.

infinite numbers (4) also take part[3] of $\mathbb{N}$. Both records, (1) and (3), are correct and do not contradict each other. They just use two different numeral systems to express $\mathbb{N}$. Traditional numeral systems do not allow us to see infinite natural numbers that we can observe now thanks to ①. Thus, we have the same object of observation – the set $\mathbb{N}$ – that can be observed by different instruments – numeral systems – with different accuracies.

Similarly, Pirahã are not able to see finite natural numbers 3, 4, and 5. In spite of the fact that Pirahã do not see them, these numbers 3, 4, and 5, belong to $\mathbb{N}$ and are visible if one uses a more powerful numeral system. Even the numeral system of Mundurukú is sufficient to observe 3, 4, and 5. Notice also that the weakness of their numeral system does not allow Pirahã to define the set (2) while Mundurukú would be able to do this.

In general, in the new methodology it is necessary always to indicate a numeral system used for computations and theoretical investigations. For instance, the words 'the set of all finite numbers' do not define a set completely in this methodology. It is always necessary to specify which instruments (numeral systems) are used to describe (and to observe) the required set and, as a consequence, to speak about 'the set of all finite numbers expressible in a fixed numeral system'. For instance, for Pirahã and Warlpiri 'the set of all finite numbers' is the set $\{1, 2\}$, for the Pitjantjatjara people 'the set of all finite numbers' is the set $\{1, 2, 3\}$ and for Mundurukú 'the set of all finite numbers' is the set $\{1, 2, 3, 4, 5\}$. We stress again that in Mathematics, as it happens in Physics, the instrument used for an observation bounds the possibility of the observation and defines the accuracy of this observation. It is not possible to say how we shall see the object of our observation if we have not clarified which instruments will be used to execute the observation.

Let us see now how one can write down different numerals expressing different infinities and infinitesimals and to execute computations with all of them. Instead of the usual symbol ∞ different infinite and/or infinitesimal numerals can be used thanks to ①. Indeterminate forms are not present and, for example, the following relations hold for infinite numbers ①, $①^2$ and $①^{-1}$, $①^{-2}$ (that are infinitesimals), as for any other (finite, infinite, or infinitesimal) number expressible in the new numeral system

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^{①} = 1, \quad 0^{①} = 0,$$

$$0 \cdot ①^{-1} = ①^{-1} \cdot 0 = 0, \quad ①^{-1} > 0, \quad ①^{-2} > 0, \quad ①^{-1} - ①^{-1} = 0,$$

$$\frac{①^{-1}}{①^{-1}} = 1, \quad (①^{-1})^0 = 1, \quad ① \cdot ①^{-1} = 1, \quad ① \cdot ①^{-2} = ①^{-1},$$

$$\frac{①^{-2}}{①^{-2}} = 1, \quad \frac{①^2}{①} = ①, \quad \frac{①^{-1}}{①^{-2}} = ①, \quad ①^2 \cdot ①^{-1} = ①, \quad ①^2 \cdot ①^{-2} = 1.$$

[3] This is a difference with respect to non-standard analysis where infinities it works with do not belong to $\mathbb{N}$.

The introduction of the numeral ① allows us to represent infinite and infinitesimal numbers in a unique framework. For this purpose a numeral system similar to traditional positional numeral systems was introduced in [24, 26]. To construct a number C in the numeral positional system with base ①, we subdivide C into groups corresponding to powers of ①:

$$C = c_{p_m}①^{p_m} + \ldots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \ldots + c_{p_{-k}}①^{p_{-k}}. \quad (5)$$

Then, the record

$$C = c_{p_m}①^{p_m} \ldots c_{p_1}①^{p_1} c_{p_0}①^{p_0} c_{p_{-1}}①^{p_{-1}} \ldots c_{p_{-k}}①^{p_{-k}} \quad (6)$$

represents the number C, where all numerals $c_i \neq 0$, they belong to a traditional numeral system and are called *grossdigits*. They express finite positive or negative numbers and show how many corresponding units $①^{p_i}$ should be added or subtracted in order to form the number C. Note that in order to have a possibility to store C in the computer memory, values k and m should be finite.

Numbers p_i in (6) are sorted in the decreasing order with $p_0 = 0$

$$p_m > p_{m-1} > \ldots > p_1 > p_0 > p_{-1} > \ldots p_{-(k-1)} > p_{-k}.$$

They are called *grosspowers* and they themselves can be written in the form (6). In the record (6), we write $①^{p_i}$ explicitly because in the new numeral positional system the number i in general is not equal to the grosspower p_i. This gives the possibility to write down numerals without indicating grossdigits equal to zero.

The term having $p_0 = 0$ represents the finite part of C since $c_0①^0 = c_0$. Terms having finite positive grosspowers represent the simplest infinite parts of C. Analogously, terms having negative finite grosspowers represent the simplest infinitesimal parts of C. For instance, the number $①^{-1} = \frac{1}{①}$ mentioned above is infinitesimal. Note that all infinitesimals are not equal to zero. In particular, $\frac{1}{①} > 0$ since it is a result of division of two positive numbers.

A number represented by a numeral in the form (6) is called *purely finite* if it has neither infinite nor infinitesimals parts. For instance, 14 is purely finite and $14 + 5.3①^{-1.5}$ is not. All grossdigits c_i are supposed to be purely finite. Purely finite numbers are used on traditional computers and for obvious reasons have a special importance for applications. All of the numbers introduced above can be grosspowers, as well, giving thus a possibility to have various combinations of quantities and to construct terms having a more complex structure.

We conclude this section by emphasizing that different numeral systems, if they have different accuracies, cannot be used together. For instance, the usage of '*many*' from the language of Pirahã in the record 5 + '*many*' has no any sense because for Pirahã it is not clear what 5 is and for people knowing what 5 is the accuracy of the answer 'many' is too low. Analogously, the records of the type $①+\omega$, $①-\aleph_0$, $①/\infty$, etc. have no sense because they include numerals developed under different methodological assumptions, in different mathematical contests, for different purposes, and, finally, numeral systems these numerals belong to have different accuracies.

Table 1. Measuring infinite sets using ①-based numerals allows one in certain cases to obtain more precise answers in comparison with the traditional cardinalities, $\aleph_0$ and $\mathcal{C}$, of Cantor.

Description of sets	Cardinality	Number of elements
the set of natural numbers $\mathbb{N}$	countable, $\aleph_0$	①
$\mathbb{N} \bigcup \{0\}$	countable, $\aleph_0$	①+1
$\mathbb{N} \setminus \{3, 5, 10, 23, 114\}$	countable, $\aleph_0$	①-5
the set of even numbers $\mathbb{E}$	countable, $\aleph_0$	$\frac{①}{2}$
the set of odd numbers $\mathbb{O}$	countable, $\aleph_0$	$\frac{①}{2}$
the set of integers $\mathbb{Z}$	countable, $\aleph_0$	2①+1
$\mathbb{Z} \setminus \{0\}$	countable, $\aleph_0$	2①
the set of square natural numbers $\mathbb{G} = \{x : x = n^2, x \in \mathbb{N}, n \in \mathbb{N}\}$	countable, $\aleph_0$	$\lfloor\sqrt{①}\rfloor$
the set of pairs of natural numbers $\mathbb{P} = \{(p, q) : p \in \mathbb{N}, q \in \mathbb{N}\}$	countable, $\aleph_0$	$①^2$
the set of numerals $\mathbb{Q}' = \{-\frac{p}{q}, \frac{p}{q} : p \in \mathbb{N}, q \in \mathbb{N}\}$	countable, $\aleph_0$	$2①^2$
the set of numerals $\mathbb{Q} = \{0, -\frac{p}{q}, \frac{p}{q} : p \in \mathbb{N}, q \in \mathbb{N}\}$	countable, $\aleph_0$	$2①^2 + 1$
the set of numerals A_2	continuum, $\mathcal{C}$	$2^{①}$
the set of numerals A'_2	continuum, $\mathcal{C}$	$2^{①} + 1$
the set of numerals A_{10}	continuum, $\mathcal{C}$	$10^{①}$
the set of numerals C_{10}	continuum, $\mathcal{C}$	$2 \cdot 10^{①}$

4 Measuring infinite sets and relations to bijections

By using the ①-based numeral system it becomes possible to measure certain infinite sets. As we have seen above, relations of the type 'many' + 1 = 'many' and $\aleph_0 - 1 = \aleph_0$ are consequences of the weakness of numeral systems applied to express numbers (finite or infinite). Thus, one of the principles of the new computational methodology consists of adopting the principle 'The part is less than the whole' to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite). Notice that this principle is a reformulation of Euclid's Common Notion 5 saying 'The whole is greater than the part'.

Let us show how, in comparison to the traditional mathematical tools used to work with infinity, the new numeral system allows one to obtain more precise answers in certain cases. For instance, Tab. 1 compares results obtained by the traditional Cantor's cardinals and the new numeral system with respect to the measure of a dozen of infinite sets (for a detailed discussion regarding the results presented in Tab. 1 and for more examples dealing with infinite sets see [18, 19, 31, 32, 41]). Notice, that in $\mathbb{Q}$ and $\mathbb{Q}'$ we calculate different numerals and not numbers. For instance, numerals $\frac{4}{1}$ and $\frac{8}{2}$ have been counted two times even though they represent the same number 4. Then, four sets of numerals having the cardinality of continuum are shown in Tab. 1 (these results are discussed more in detail in the next section). Among them we denote by A_2 the set of numbers $x \in [0, 1)$ expressed in the binary positional numeral system, by A'_2 the set being the same as A_2 but with x belonging to the closed interval $[0, 1]$, by A_{10} the set of numbers $x \in [0, 1)$ expressed in the decimal positional numeral system, and finally we have the set $C_{10} = A_{10} \cup B_{10}$, where B_{10} is the set of numbers $x \in [1,2)$ expressed in the decimal positional numeral system. It is worthwhile to notice also that grossone-based numbers from Tab. 1 can be ordered as follows

$$\lfloor\sqrt{①}\rfloor < \frac{①}{2} < ① - 5 < ① < 2① < 2① + 1 <$$

$$①^2 < 2①^2 + 1 < 2^{①} < 2^{①} + 1 < 10^{①} < 2 \cdot 10^{①}.$$

It can be seen from Tab. 1 that Cantor's cardinalities say only whether a set is countable or uncountable while the ①-based numerals allow us to express the exact number of elements of the infinite sets. However, both numeral systems – the new one and the numeral system of infinite cardinals – do not contradict one another. Both Cantor's numeral system and the new one give correct answers, but their answers have *different accuracies*. By using an analogy from physics we can say that the lens of our new 'telescope' used to observe infinities and infinitesimals is stronger and where Cantor's 'telescope' allows one to distinguish just two dots (countable sets and the continuum) we are able to see many different dots (infinite sets having different number of elements).

The ①-base numeral system, as all numeral systems, cannot express all numbers and give answers to all questions. Let us consider, for instance, the set of *extended natural numbers* indicated as $\widehat{\mathbb{N}}$ and including $\mathbb{N}$ as a proper subset

$$\widehat{\mathbb{N}} = \{\underbrace{1, 2, \ldots, ① - 1, ①}_{\text{Natural numbers}}, ① + 1, ① + 2, \ldots, 2① - 1, 2①, 2① + 1, \ldots$$
$$①^2 - 1, ①^2, ①^2 + 1, \ldots 3①^{①} - 1, 3①^{①}, 3①^{①} + 1, \ldots\}. \quad (7)$$

What can we say with respect to the number of elements of the set $\widehat{\mathbb{N}}$? The introduced numeral system based on ① is too weak to give an answer to this question. It is necessary to introduce in a way a more powerful numeral system by defining new numerals (for instance, ②, ③, etc).

In order to see how the principle 'The part is less than the whole' agrees with traditional views on infinite sets, let us consider two illustrative examples. The first of them is related to the one-to-one correspondence that can be established between the sets of natural and odd numerus. Namely, odd numbers can be put in a one-to-one correspondence with all natural numbers in spite of the fact that $\mathbb{O}$ is a proper subset of $\mathbb{N}$

$$\begin{array}{ll} \text{odd numbers:} & 1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots \\ & \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow \\ \text{natural numbers:} & 1,\ 2,\ 3,\ 4\ 5,\ 6,\ \ldots \end{array} \quad (8)$$

The usual conclusion is that both sets are countable and they have the same cardinality $\aleph_0$.

Let us see now what we can say from the new methodological positions. We know now that when one executes the operation of counting, the accuracy of the result depends on the numeral system used for counting. Proposing to Pirahã to measure sets consisting of four apples and five apples would give us the answer that both sets of apples have many elements. This answer is correct but its precision is low due to the weakness of the numeral system used to measure the sets.

Thus, the introduction of the notion of accuracy for measuring sets is very important and should be applied for infinite sets also. Since for cardinal numbers it follows

$$\aleph_0 + 1 = \aleph_0, \qquad \aleph_0 + 2 = \aleph_0, \qquad \aleph_0 + \aleph_0 = \aleph_0,$$

these relations suggest that the accuracy of the cardinal numeral system of Alephs an is not sufficiently high to see the difference with respect to the number of elements of the two sets from (8).

In order to look at the record (8) using the new numeral system we need the following fact from [24]: the sets of even and odd numbers have ①/2 elements each and, therefore, ① is even. It is also necessary to remind that numbers that are larger than ① are not natural, they are extended natural numbers. For instance, $① + 1$ is odd but not natural, it is extended natural, see (7). Thus, the last odd natural number is ①-1. Since the number of elements of the set of odd numbers is equal to $\frac{①}{2}$, we can write down not only initial (as it is usually done traditionally) but also the final part of (8)

$$\begin{array}{llllllllll} 1, & 3, & 5, & 7, & 9, & 11, & \ldots & ① - 5, & ① - 3, & ① - 1 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4 & 5, & 6, & \ldots & \frac{①}{2} - 2, & \frac{①}{2} - 1, & \frac{①}{2} \end{array} \qquad (9)$$

concluding so (8) in a complete accordance with the principle ‘The part is less than the whole’. Both records, (8) and (9), are correct but (9) is more accurate, since it allows us to observe the final part of the correspondence that is invisible if (8) is used.

The accuracy of the ①-based numeral system allows us to measure also, for instance, such sets as $\mathbb{O}' = \mathbb{O}\backslash\{3\}$ and $\mathbb{O}'' = \mathbb{O}\backslash\{1, ① - 1\}$. The set $\mathbb{O}'$ is constructed by excluding one element from $\mathbb{O}$ and the set $\mathbb{O}''$ by excluding from $\mathbb{O}$ two elements. Thus, $\mathbb{O}'$ and $\mathbb{O}''$ have $\frac{①}{2}-1$ and $\frac{①}{2}-2$ elements, respectively. In case one wishes to establish the corresponding bijections, starting with natural numbers $1, 2, 3, \ldots$ we obtain for these two sets

$$\begin{array}{llllllllll} 1, & 5, & 7, & 9, & 11, & 13, & \ldots & ① - 5, & ① - 3, & ① - 1 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4 & 5, & 6, & \ldots & \frac{①}{2} - 3, & \frac{①}{2} - 2, & \frac{①}{2} - 1 \end{array} \qquad (10)$$

$$\begin{array}{llllllllll} 3, & 5, & 7, & 9, & 11, & 13, & \ldots & ① - 7, & ① - 5, & ① - 3 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow \\ 1, & 2, & 3, & 4 & 5, & 6, & \ldots & \frac{①}{2} - 4, & \frac{①}{2} - 3, & \frac{①}{2} - 2 \end{array} \qquad (11)$$

In order to become more familiar with natural and extended natural numbers let us consider one more example where we multiply each element of the set of natural numbers, $\mathbb{N}$, by 2. We would like to study the resulting set, that is called $\mathbb{E}^2$ hereinafter, to calculate the number of its elements, and to specify which among its elements are natural and which ones are extended natural numbers and how many they are.

The introduction of the new numeral system allows us to write down the set, $\mathbb{N}$, of natural numbers in the form (7). By definition, the number of elements of $\mathbb{N}$ is equal to ①. Thus, after multiplication of each of the elements of $\mathbb{N}$ by 2, the resulting set, $\mathbb{E}^2$, will also have grossone elements. In particular, the number $\frac{①}{2}$ multiplied by 2 gives us ① and $\frac{①}{2} + 1$ multiplied by 2 gives us $① + 2$ that is even extended natural number, see (7). Analogously, the last element of $\mathbb{N}$, i.e., ①, multiplied by 2 gives us 2①. Thus, the set of even numbers $\mathbb{E}^2$ can be written as follows

$$\mathbb{E}^2 = \{2, 4, 6, \ \ldots \ ① - 4, ① - 2, ①, ① + 2, ① + 4, \ \ldots \ 2① - 4, 2① - 2, 2①\},$$

where numbers $\{2, 4, 6, \ \ldots \ ① - 4, ① - 2, ①\}$ are even and natural (they are $\frac{①}{2}$) and numbers $\{① + 2, ① + 4, \ \ldots \ 2① - 4, 2① - 2, 2①\}$ are even and extended natural, they also are $\frac{①}{2}$.

5 Turing machines and infinite sequences

In this section we present some results related to Turing machines with infinite tapes (the presentation has been simplified, see [41, 42] for a comprehensive discussion). Traditionally, an *infinite sequence* $\{a_n\}, a_n \in A$, $n \in \mathbb{N}$, is defined as a function having the set of natural numbers, $\mathbb{N}$, as the domain and a set A as the codomain. A *subsequence* $\{b_n\}$ is defined as a sequence $\{a_n\}$ from which some of its elements have been removed. In spite of the fact that the removal of the elements from $\{a_n\}$ can be directly observed, the traditional point of view on sequences does not allow one to register, in the case where the obtained subsequence $\{b_n\}$ is infinite, the fact that $\{b_n\}$ has less elements than the original infinite sequence $\{a_n\}$.

Let us study what happens when the new approach is used. The definition of infinite sequences should be done more precise in a complete analogy to finite sequences. In the finite case, to define a sequence $a_1, a_2, \ldots, a_n$ the number, n, of its elements should be explicitly declared. Thanks to the introduction of ①-based numerals we are able to express infinite numbers, as well and, as a consequence, we extend this definition directly to the infinite case, i.e., to define an infinite sequence $a_1, a_2, \ldots, a_n$ its infinite number of elements, n, should be provided.

Since the new numeral system allows us to express the number of elements of the set $\mathbb{N}$ as ① and due to the sequence definition given above, any sequence having $\mathbb{N}$ as the domain has ① elements. Such sequences are called *complete.* Notice that, among other things, this definition states that there cannot exist infinite sequences having more than ① elements. However, since we can express infinite integers less than ①, infinite sequences having less than ① elements can exist and can be described using ①-based numerals. In fact, the notion of subsequence is introduced as a sequence from which some of its elements have been removed. This means that the resulting subsequence will have less elements than the original sequence and the infinite number of elements of infinite subsequences can be expressed.

For instance, let us consider two infinite sequences: $\{a_n\}$ and $\{b_n\}$. The first sequence $\{a_n\}$, $1 \leq n \leq ①$, with $a_n = n - 1$. This sequence has ① elements and it is, therefore, complete. Its first element is $a_1 = 0$ and its last element is $a_① = ① - 1$. The second infinite sequence, $\{b_n\}$, that is a subsequence of the first one is defined as follows: $\{b_n\}$, $1 \leq n \leq 0.5①$, with $b_n = n - 1$. Thus, both sequences, $\{a_n\}$ and $\{b_n\}$, have the same general element, $a_n = b_n = n - 1$, the same first element, $a_1 = b_1 = 0$, and both are infinite but the first sequence is complete and the second one is not since it has $0.5① < ①$ elements and its last element is $b_{0.5①} = 0.5① - 1$.

Suppose now that we have a Turing machine with an infinite tape that contains an output written using symbols $\{0, 1, \ldots b-2, b-1\}$ with a finite b. The traditional point of view allows us to distinguish neither tapes having different infinite lengths nor machines using different alphabets, i.e., $\{0, 1, \ldots B-2, B-1\}$ with $B \neq b$. The question of the possibility to have different infinite tapes is not discussed and it is supposed that machines with all output alphabets have the same computational power if their tapes are infinite. This happens because the traditional numeral systems used to describe Turing machines do not allow us to see these differences. The new numeral system offers such a possibility giving a chance to describe Turing machines in a more precise way and to distinguish them at infinity.

In the new framework, it is not sufficient to say that the tape is infinite. It is necessary to define the infinite length of the tape explicitly. As an example, let us consider a Turing machine having the tape ① positions long. Output sequences are written on the tape using symbols from an output alphabet, let it be again $\{0, 1, \ldots b-2, b-1\}$ with a finite b. The importance of the discussion on the infinite sequences provided above for Turing machines becomes clear now: the output sequences of symbols, as all sequences, though infinite cannot have more than ① elements.

Moreover, we can make a more accurate analysis and count the precise number of infinite output sequences of symbols that the machine can produce. It is obvious that its outputs can be viewed as numerals in the positional numeral system with the finite radix b

$$(a_1 a_2 \ldots a_{①-1} a_①)_b, \qquad a_i \in \{0, 1, \ldots b-2, b-1\}, \;\; 1 \leq i \leq ①. \tag{12}$$

This means that we have ① positions that can be filled in with b symbols each, i.e., this machine called hereinafter T_1 can produce $b^①$ different outputs. Then, if we consider another machine, T_2, having the tape with ①-1 positions and outputs written using the same base, b, the number of its outputs is $b^{①-1} < b^①$ and each of them is one position shorter than outputs of T_1. Moreover, if we consider the third machine, T_3, having the tape with ① positions and outputs written using a base $B > b$, the number of its outputs is $B^① > b^①$. In other words, the machine T_3 is more powerful then the machine T_1 that, in its turn, is more powerful than the machine T_2.

Let us give a couple of illustrations. We start by considering a Turing machine T_4 working with the alphabet $\{0, 1, 2\}$, the tape with ①/2 positions, and computing the following output

$$\underbrace{0, 1, 2, 0, 1, 2, 0, 1, 2, \ \dots \ 0, 1, 2, 0, 1, 2}_{\text{①/2 positions}}. \tag{13}$$

Then a Turing machine T_5 working with the output alphabet $\{0, 1\}$ and the tape with ①/2 positions cannot produce a sequence of symbols computing (13). In fact, since the numeral 2 does not belong to the alphabet $\{0, 1\}$ it should be coded by more than one symbol. One of codifications using the minimal number of symbols in the alphabet $\{0, 1\}$ necessary to code numbers $0, 1, 2$ is $\{00, 01, 10\}$. Then the output corresponding to (13) and computed in this codification should be

$$00, 01, 10, 00, 01, 10, 00, 01, 10, \ \dots \ 00, 01, 10, 00, 01, 10. \tag{14}$$

Since the output (13) contains ①/2 positions, the output (14) should contain ① positions. However, by the definition of T_5 it can produce outputs that have only ①/2 positions.

Let us consider now a Turing machine T_6 working with the alphabet $\{0, 1, 2\}$ as T_4 but the infinite tape of T_6 is one position longer than the tape of T_4, i.e., it has ①/2+1 positions, and T_6 computes the following output

$$\underbrace{0, 1, 2, 0, 1, 2, 0, 1, 2, \ \dots \ 0, 1, 2, 0, 1, 2, 0}_{\text{①/2+1 positions}}. \tag{15}$$

Then there is no a Turing machine working with the output alphabet $\{0, 1\}$ and coding the numbers $0, 1, 2$ as $\{00, 01, 10\}$ such that it is able to compute the output corresponding to (15) in this codification. The proof is very easy and is based on the fact that infinite sequences cannot have more than ① elements. Since the output (15) contains ①/2+1 positions, the output

$$00, 01, 10, 00, 01, 10, 00, 01, 10, \ \dots \ 00, 01, 10, 00, 01, 10, 00.$$

should contain ① + 2 positions. However, infinite sequences cannot have more than ① elements. Notice that significantly more sophisticated results for deterministic and non-deterministic Turing machines can be found in [41–43].

6 Concluding remarks

In this paper infinite sets and Turing machines with different infinite tapes have been studied using a recently introduced positional numeral system with the infinite radix ①. It has been shown that in certain cases the new numerals allow one to obtain more precise results in dealing with infinite quantities in comparison to numeral systems traditionally used for this purpose.

In particular, the following observation (see [31] for a detailed discussion) can be made for the set C_b^k of numerals expressible in the positional numeral system with the finite radix b and k digits $\{0, 1, \ldots b-2, b-1\}$ where k is infinite

$$(a_1 a_2 \ldots a_{k-1} a_k)_b, \qquad a_i \in \{0, 1, \ldots b-2, b-1\}, \;\; 1 \leq i \leq k. \tag{16}$$

Clearly, this is a simple generalization of the record (12) where we have $k = ①$. Analogously to the analysis made above it follows that the number of numerals expressible in the system (16) is b^k and for infinite values of k the set C_b^k should have the cardinality of continuum in the traditional language. Let us consider now $k_1 = \lfloor \log_b ① \rfloor$ where $\lfloor x \rfloor$ is the integer part of x. Note that k_1 is infinite since ① is infinite. It follows then that

$$b^{\lfloor \log_b ① \rfloor} < b^{\log_b ①} = ①,$$

i.e., with respect to the traditional language the set $C_b^{\log_b ①}$ would be countable. Analogously, many different instances of infinite sets that are constructed starting from the continuum framework and resulting at the end to be countable can be exhibited. For example, for infinite $k = 3\lfloor \log_b ① \rfloor$ and $k = 0.5\lfloor \log_b ① \rfloor$ it follows that

$$b^{3\lfloor \log_b ① \rfloor} < b^{3 \log_b ①} = ①^3 < b^{①}, \qquad b^{0.5\lfloor \log_b ① \rfloor} < b^{0.5 \log_b ①} = \sqrt{①} < b^{①},$$

i.e., the sets $C_b^{3 \log_b ①}$ and $C_b^{0.5 \log_b ①}$ would be also countable from the traditional point of view.

Thus, the ①-based numeral system allows us to distinguish new infinite sets that were invisible using traditional instruments both within continuum and numerable sets. Thanks to the ①-based numerals it becomes possible to calculate the exact number of elements of old (see Table 1) and new sets and to exhibit sets that were constructed as continuum but are indeed countable bridging so the gap between the two groups of sets (see [31] for a detailed discussion). This fact, among other things, allows us to see that the computational power of Turing machines with different infinite tapes is different. Reminding our example with the microscope we are able now to see instead of two dots (countable and continuum) many different dots.

In this paper only two applications where ①-based numerals are useful have been discussed: infinite sets and Turing machines. More examples showing how these numerals can be successfully used can be found in the following publications: Euclidean and hyperbolic geometry (see [20, 21]), percolation (see [12, 13, 44]), fractals (see [25, 27, 35, 44]), infinite series and the Riemann zeta function (see [29, 34, 46]), the first Hilbert problem and lexicographic ordering (see [31, 41–43, 39]), cellular automata (see [5–7]).

In particular, numerical computations with infinities and infinitesimals expressed by ①-based numerals are discussed in the following papers: numerical differentiation, solutions of systems of linear equations, and optimization (see [4, 28, 33, 47]), ordinary differential equations (see [37, 38]).

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