

# On accuracy of mathematical languages used to deal with the Riemann zeta function and the Dirichlet eta function

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## Abstract

The Riemann Hypothesis has been of central interest to mathematicians for a long time and many unsuccessful attempts have been made to either prove or disprove it. Since the Riemann zeta function is defined as a sum of the infinite number of items, in this paper, we look at the Riemann Hypothesis using a new applied approach to infinity allowing one to easily execute numerical computations with various infinite and infinitesimal numbers in accordance with the principle ‘The part is less than the whole’ observed in the physical world around us. The new approach allows one to work with functions and derivatives that can assume not only finite but also infinite and infinitesimal values and this possibility is used to study properties of the Riemann zeta function and the Dirichlet eta function. A new computational approach allowing one to evaluate these functions at certain points is proposed. Numerical examples are given. It is emphasized that different mathematical languages can be used to describe mathematical objects with different accuracies. The traditional and the new approaches are compared with respect to their application to the Riemann zeta function and the Dirichlet eta function. The accuracy of the obtained results is discussed in detail.

**Key Words:** Infinite and infinitesimal numbers and numerals; accuracy of mathematical languages; Riemann zeta function; Dirichlet eta function; divergent series.

## 1 Introduction

The Riemann zeta function

$$\zeta(s) = \sum_{u=1}^{\infty} \frac{1}{u^s} \quad (1)$$

defined for complex numbers  $s$  is one of the most important mathematical objects discovered so far (see [4, 7]). It has been first introduced and studied by Euler (see

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[8, 9, 10, 11]) who proved the famous identity

$$\sum_{u=1}^{\infty} \frac{1}{u^s} = \prod_{p \text{ primes}} \frac{1}{1 - p^{-s}} \quad (2)$$

being one of the strongest sources of the interest to the Riemann zeta function. Many interesting results have been established for this function (see [4, 7] for a complete discussion). In the context of this paper, the following results will be of our primary interest.

It is known that  $\zeta(s)$  diverges on the open half-plane of  $s$  such that the real part  $\Re(s) < 1$ . Then, for  $s = 0$  the following value is attributed to  $\zeta(0)$ :

$$\zeta(0) = \sum_{u=1}^{\infty} \frac{1}{u^0} = 1 + 1 + 1 + \dots = -\frac{1}{2}. \quad (3)$$

It is also known that the Riemann zeta function has the following relation

$$\eta(s) = (1 - 2^{1-s})\zeta(s) \quad (4)$$

to the Dirichlet eta function

$$\eta(s) = \sum_{u=1}^{\infty} \frac{(-1)^{u-1}}{u^s}. \quad (5)$$

It has been shown for the Riemann zeta function that it has trivial zeros at the points  $-2, -4, \dots$ . It is also known that any non-trivial zero lies in the complex set  $\{s : 0 < \Re(s) < 1\}$ , called the critical strip. The complex set  $\{s : \Re(s) = 1/2\}$  is called the critical line. The Riemann Hypothesis asserts that any non-trivial zero  $s$  has the real part  $\Re(s) = 1/2$ , i.e., lies on the critical line.

This problem has been of central interest to mathematicians for a long time and many unsuccessful attempts have been made to either prove or disprove it. Since both the Riemann zeta function and the Dirichlet eta function are defined as sums of the infinite number of items, in this paper, we look at them and at the Riemann Hypothesis using a new approach to infinity introduced in [15, 16, 17, 20, 21]. This approach, in particular, incorporates the following two ideas.

i) The point of view on infinite and infinitesimal quantities applied in this paper uses strongly two methodological ideas borrowed from the modern Physics: relativity and interrelations holding between the object of an observation and the tool used for this observation. The latter is directly related to interrelations between numeral systems<sup>1</sup> used to describe numbers and numbers (and other mathematical objects) themselves. Numerals that we use to write down numbers, functions, etc.

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<sup>1</sup>We remind that *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '3', 'three', and 'III' are different numerals, but they all represent the same number.

are among our tools of investigation and, as a result, they strongly influence our capabilities to study mathematical objects.

ii) Both standard and non-standard Analysis study mainly functions assuming finite values. In [17, 19], functions and their derivatives that can assume finite, infinite, and infinitesimal values and can be defined over finite, infinite, and infinitesimal domains have been studied. Infinite and infinitesimal numbers are not auxiliary entities in the new approach, they are full members in it and can be used in the same way as finite constants.

In the next section, we present very briefly the new methodology for treating infinite and infinitesimal quantities indicating mainly the facts that are used directly in this paper. A comprehensive introduction and numerous examples of its usage can be found in [17, 21] while some applications can be found in [5, 6, 14, 17, 18, 19, 22, 23].

## **2 Accuracy of mathematical languages and a new numeral system for dealing with infinity**

In order to introduce the new methodology, let us consider a study published in *Science* by Peter Gordon (see [12]) where he describes a primitive tribe living in Amazonia – Pirahã – that uses a very simple numeral system for counting: one, two, many. For Pirahã, all quantities larger than two are just ‘many’ and such operations as  $2+2$  and  $2+1$  give the same result, i.e., ‘many’. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, and 5, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that.

It is important to emphasize that the records  $2 + 1 = \text{‘many’}$  and  $2 + 2 = \text{‘many’}$  are correct in their language and if one is satisfied with the accuracy of the answer ‘many’, it can be used (and *is used* by Pirahã) in practice. Note that also for us, people knowing that  $2 + 1 = 3$  and  $2 + 2 = 4$ , the result of Pirahã is not wrong, it is just *inaccurate*. Thus, if one needs a more precise result than ‘many’, it is necessary to introduce a more powerful mathematical language (a numeral system in this case) allowing one to express the required answer in a more accurate way. By using modern powerful numeral systems where additional numerals for expressing the numbers ‘three’ and ‘four’ have been introduced, we can notice that within ‘many’ there are several objects and the numbers 3 and 4 are among these unknown to Pirahã) objects.

Thus, the choice of the mathematical language depends on the practical problem that is to be solved and on the accuracy required for such a solution. In dependence of this accuracy, a numeral system that would be able to express the numbers composing the answer should be chosen.

Such a situation is typical for natural sciences where it is well known that instruments bound and influence results of observations. When physicists see a

black dot in their microscope they cannot say: The object of the observation *is* the black dot. They are obliged to say: the lens used in the microscope allows us to see the black dot and it is not possible to say anything more about the nature of the object of the observation until we change the instrument - the lens or the microscope itself - by a more precise one. Then, probably, when a stronger lens will be put in the microscope, physicists will be able to see that the object that seemed to be one dot with the new lens can be viewed as, e.g., two dots.

Note that both results (one dot and two dots) correctly represent the reality with the accuracy of the chosen instrument of the observation (lens). Physicists decide the level of the precision they need and obtain a result depending of the chosen level of the accuracy. In the moment when they put a lens in the microscope, they have decided the minimal (and the maximal) size of objects that they will be able to observe. If they need a more precise or a more rough answer, they change the lens of their microscope.

Analogously, when mathematicians have decided which mathematical languages (in particular, which numeral systems) they will use in their work, they have decided which mathematical objects they will be able to observe and to describe.

In natural sciences there always exists the triad – the researcher, the object of investigation, and tools used to observe the object – and the instrument used to observe the object bounds and influences results of observations. The same happens in Mathematics studying numbers and objects that can be constructed by using numbers. Numeral systems used to express numbers are instruments of observations used by mathematicians. The usage of powerful numeral systems gives the possibility of obtaining more precise results in Mathematics in the same way as usage of a good microscope gives the possibility to obtain more precise results in Physics. Let us now return to Pirahã and make the first relevant observation with respect to the subject of this paper.

The weakness of the numeral system of Pirahã gives them inaccurate answers to arithmetical operations with finite numbers where modern numeral systems are able to provide exact solutions. However, Pirahã obtain also such results as

$$\text{'many'} + 1 = \text{'many'}, \quad \text{'many'} + 2 = \text{'many'}, \quad (6)$$

which are very familiar to us in the context of views on infinity used in the traditional calculus where the famous numeral  $\infty$  is used:

$$\infty + 1 = \infty, \quad \infty + 2 = \infty. \quad (7)$$

This observation leads us to the following idea: *Probably our difficulties in working with infinity in general (and in particular, with the the Riemann zeta function) is not connected to the nature of infinity itself but is a result of inadequate numeral systems that we use to work with infinity, more precisely, to express infinite numbers.*

Recently, a new numeral system has been developed in order to express finite, infinite, and infinitesimal numbers in a unique framework (a rather comprehensive

description of the new methodology can be found in [17, 21]). The main idea consists of measuring infinite and infinitesimal quantities by different (infinite, finite, and infinitesimal) units of measure. This is done by using a new numeral.

An infinite unit of measure has been introduced for this purpose in [17, 21] as the number of elements of the set  $\mathbb{N}$  of natural numbers. It is expressed by a new numeral  $\textcircled{1}$  called *grossone*. It is necessary to emphasize immediately that the infinite number  $\textcircled{1}$  is neither Cantor's  $\aleph_0$  nor  $\omega$  and the new approach is not related to the non-standard analysis.

One of the important differences with respect to traditional views consists of the fact that the new approach continuously emphasizes the distinction between numbers and numerals and studies in depth numeral systems. In general, neither standard analysis nor non-standard one pay a lot of attention to the necessity, in order to be able to execute an arithmetical operation, to have on hand numerals required to express both the operands and the result of the operation. The accuracy of the obtained result (i.e., how well the numeral chosen to represent the result of the operation expresses the resulting quantity) is not studied either. For the approach introduced in [17, 21] both issues are crucial. Since for any fixed numeral system  $S$  there exist operations that cannot be executed in  $S$  because  $S$  has no suitable numerals, these marginal situations are investigated in detail. We can remind Pirahã in this occasion (see (6), (7)) but even the modern powerful positional numeral systems which are not able to work with numerals having, e.g.,  $10^{100}$  digits. It is shown in [17, 20, 21, 23] that very often the margins of the expressibility of numeral systems are connected directly to interesting research problems.

Another important peculiarity of the new approach consists of the fact that  $\textcircled{1}$  has both cardinal and ordinal properties as usual finite natural numbers have. In fact, infinite positive integers that can be viewed through numerals including grossone can be interpreted in the terms of the number of elements of certain infinite sets. For instance, the set of even numbers has  $\frac{\textcircled{1}}{2}$  elements and the set of integers has  $2\textcircled{1}+1$  elements. Thus, the new numeral system allows one to distinguish within countable sets many different sets having the different infinite number of elements (remember again the analogy with the microscope). Analogously, within uncountable sets it is possible to distinguish sets having, for instance,  $2^{\textcircled{1}}$  elements,  $10^{\textcircled{1}}$  elements, and even  $\textcircled{1}^{\textcircled{1}} - 1$ ,  $\textcircled{1}^{\textcircled{1}}$ , and  $\textcircled{1}^{\textcircled{1}} + 1$  elements and to show (see [20, 21]) that

$$\frac{\textcircled{1}}{2} < \textcircled{1} < 2\textcircled{1}+1 < 2^{\textcircled{1}} < 10^{\textcircled{1}} < \textcircled{1}^{\textcircled{1}} - 1 < \textcircled{1}^{\textcircled{1}} < \textcircled{1}^{\textcircled{1}} + 1.$$

Formally, grossone is introduced as a new number by describing its properties postulated by the *Infinite Unit Axiom* (see [17, 21]). This axiom is added to axioms for real numbers similarly to addition of the axiom determining zero to axioms of natural numbers when integers are introduced. Inasmuch as it has been postulated that grossone is a number, all other axioms for numbers hold for it, too. Particularly, associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, existence of inverse elements with respect to addition and multiplication hold for grossone as for finite numbers.

This means that the following relations hold for grossone, as for any other number

$$0 \cdot \textcircled{1} = \textcircled{1} \cdot 0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \frac{\textcircled{1}}{\textcircled{1}} = 1, \quad \textcircled{1}^0 = 1, \quad 1^{\textcircled{1}} = 1, \quad 0^{\textcircled{1}} = 0. \quad (8)$$

Various numeral systems including  $\textcircled{1}$  can be used for expressing infinite and infinitesimal numbers. In particular, records similar to traditional positional numeral systems can be used (see [17, 18]). In order to construct a number  $C$  in this system, we subdivide  $C$  into groups corresponding to powers of grossone:

$$C = c_{p_m} \textcircled{1}^{p_m} + \dots + c_{p_1} \textcircled{1}^{p_1} + c_{p_0} \textcircled{1}^{p_0} + c_{p_{-1}} \textcircled{1}^{p_{-1}} + \dots + c_{p_{-k}} \textcircled{1}^{p_{-k}}. \quad (9)$$

Then, the record

$$C = c_{p_m} \textcircled{1}^{p_m} \dots c_{p_1} \textcircled{1}^{p_1} c_{p_0} \textcircled{1}^{p_0} c_{p_{-1}} \textcircled{1}^{p_{-1}} \dots c_{p_{-k}} \textcircled{1}^{p_{-k}} \quad (10)$$

represents the number  $C$ , where finite numbers  $c_i \neq 0$  called *grossdigits* can be positive or negative. They show how many corresponding units should be added or subtracted in order to form the number  $C$ . Grossdigits can be expressed by several symbols.

Numbers  $p_i$  in (10) called *grosspowers* can be finite, infinite, and infinitesimal, they are sorted in the decreasing order with  $p_0 = 0$

$$p_m > p_{m-1} > \dots > p_1 > p_0 > p_{-1} > \dots > p_{-(k-1)} > p_{-k}.$$

In the record (10), we write  $\textcircled{1}^{p_i}$  explicitly because in the new numeral positional system the number  $i$  in general is not equal to the grosspower  $p_i$ .

Finite numbers in this new numeral system are represented by numerals having only one grosspower  $p_0 = 0$ . In fact, if we have a number  $C$  such that  $m = k = 0$  in representation (10), then due to (8), we have  $C = c_0 \textcircled{1}^0 = c_0$ . Thus, the number  $C$  in this case does not contain grossone and is equal to the grossdigit  $c_0$  being a conventional finite number expressed in a traditional finite numeral system.

The simplest infinitesimal numbers are represented by numerals  $C$  having only finite or infinite negative grosspowers, e.g.,  $21.3 \textcircled{1}^{-243.65} 5 \textcircled{1}^{-154}$ . Then, the infinitesimal number  $\frac{1}{\textcircled{1}} = \textcircled{1}^{-1}$  is the inverse element with respect to multiplication for  $\textcircled{1}$ :

$$\textcircled{1}^{-1} \cdot \textcircled{1} = \textcircled{1} \cdot \textcircled{1}^{-1} = 1. \quad (11)$$

Note that all infinitesimals are not equal to zero. Particularly,  $\frac{1}{\textcircled{1}} > 0$  because it is a result of division of two positive numbers.

The simplest infinite numbers are expressed by numerals having positive finite or infinite grosspowers without infinitesimals. They have infinite parts and can also have a finite part and infinitesimal ones. For instance, the number

$$31.5 \textcircled{1}^{14.8} (-0.645) \textcircled{1}^5 7.89 \textcircled{1}^0 37 \textcircled{1}^{-4.29} 72.8 \textcircled{1}^{-360.21}$$

has two infinite parts  $31.5 \textcircled{1}^{14.8}$  and  $-0.645 \textcircled{1}^5$  one finite part  $7.89 \textcircled{1}^0$  and two infinitesimal parts  $37 \textcircled{1}^{-4.29}$  and  $72.8 \textcircled{1}^{-360.21}$ . All of the numbers introduced above can be used as grosspowers, as well, giving so a possibility to have various combinations of quantities and to construct terms having a more complex structure. In this paper, we are interested only on numbers having integer grosspowers.

### 3 Sums having a fixed infinite number of items and integrals over fixed infinite domains

Introduction of the new numeral,  $\textcircled{1}$ , allows us to distinguish within  $\infty$  many different infinite numbers in the same way as the introduction of modern numeral systems used to express finite numbers allows us to distinguish numbers ‘three’ and ‘four’ within ‘many’.

Thus, the Aristotle principle ‘The part is less than the whole’ leading to the fact that  $x + 1 > x$  can be applied to all numbers  $x$  (finite, infinite, and infinitesimal) and not only to finite numbers as it is usually done in the traditional mathematics (this point is discussed in detail in [17, 21]). In general, we have that  $x + y > x$ ,  $y > 0$ , where both  $y$  and  $x$  can be finite, infinite, or infinitesimal. For instance, it follows

$$\textcircled{1} + 1 > \textcircled{1}, \quad 1 + \textcircled{1}^{-1} > 1, \quad \textcircled{1}^2 + 1 > \textcircled{1}^2, \quad \textcircled{1}^2 + \textcircled{1}^{-1} > \textcircled{1}^2.$$

This means that such records as  $S = a_1 + a_2 + \dots$  or  $\sum_{i=1}^{\infty} a_i$  become unprecise in the new mathematical language using grossone. By continuation the analogy with Pirahã (see (6), (7)), the record  $\sum_{i=1}^{\infty} a_i$  becomes a kind of  $\sum_{i=1}^{\text{many}} a_i$ .

It is worthwhile noticing also that the symbol  $\infty$  cannot be used in the same expression with numerals using  $\textcircled{1}$  (analogously, the record ‘many’+4 has no sense because it uses symbols from two different languages having different accuracies).

As a consequence, if one wants to work with an infinite number,  $n$ , of items in a sum  $\sum_{i=0}^n a_i$  then it is necessary to fix explicitly this number using numerals available for expressing infinite numbers in a chosen numeral system (e.g., the numeral system introduced in the previous section can be taken for this purpose). The same situation we have with sums having a finite number of items: it is not sufficient to say that the number,  $n$ , of items in the sum is finite, it is necessary to fix explicitly the value of  $n$  using for this purpose numerals available in a chosen traditional numeral system to express finite numbers.

The new numeral system using grossone allows us to work with expressions involving infinite numbers and to obtain, where it is possible, as results infinite, finite, and infinitesimal numbers. Obviously, this is of a particular interest in connection with the Riemann zeta function. For instance, it becomes possible to reconsider the result (3) that is very difficult to be fully realized by anyone who is not a mathematician. In fact, when one has an infinite sum of positive finite integers he or she would expect to have an infinite positive integer as a result. In contrast, (3) proposes a negative finite fractional number as the answer.

In order to become acquainted with the way the new methodology is applied to the theory of divergent series, let us consider several example. We start by considering two infinite series  $S_1 = 7 + 7 + 7 + 7 + \dots$  and  $S_2 = 3 + 3 + 3 + 3 + \dots$ . The traditional analysis gives us a very poor answer that both of them diverge to infinity. Such operations as, e.g.,  $\frac{S_2}{S_1}$  and  $S_2 - S_1$  are not defined.

The new mathematical language using grossone allows us to indicate explicitly

the number of their items. Suppose that the series  $S_1$  has  $k$  items and  $S_2$  has  $n$  items:

$$S_1(k) = \underbrace{7+7+7+\dots+7}_{k \text{ items}}, \quad S_2(n) = \underbrace{3+3+3+\dots+3}_{n \text{ items}}.$$

Then  $S_1(k) = 7k$  and  $S_2(n) = 3n$  and by giving different numerical values (finite or infinite) to  $k$  and  $n$  we obtain different numerical values for the sums. For chosen  $k$  and  $n$  it becomes possible to calculate  $S_2(n) - S_1(k)$  (analogously, the expression  $\frac{S_1(k)}{S_2(n)}$  can be calculated). If, for instance,  $k = 5\mathbb{1}$  and  $n = \mathbb{1}$  we obtain  $S_1(5\mathbb{1}) = 35\mathbb{1}$ ,  $S_2(\mathbb{1}) = 3\mathbb{1}$  and it follows

$$S_2(\mathbb{1}) - S_1(5\mathbb{1}) = 3\mathbb{1} - 35\mathbb{1} = -32\mathbb{1} < 0.$$

If  $k = 3\mathbb{1}$  and  $n = 7\mathbb{1} + 2$  we obtain  $S_1(3\mathbb{1}) = 21\mathbb{1}$ ,  $S_2(7\mathbb{1} + 2) = 21\mathbb{1} + 6$  and it follows

$$S_2(7\mathbb{1} + 2) - S_1(3\mathbb{1}) = 21\mathbb{1} + 6 - 21\mathbb{1} = 6.$$

It becomes also possible to calculate sums having an infinite number of infinite or infinitesimal items. Let us consider, for instance, sums

$$S_3(l) = \underbrace{2\mathbb{1} + 2\mathbb{1} + \dots + 2\mathbb{1}}_{l \text{ items}}, \quad S_4(m) = \underbrace{4\mathbb{1}^{-1} + 4\mathbb{1}^{-1} + \dots + 4\mathbb{1}^{-1}}_{m \text{ items}}.$$

For  $l = m = 0.5\mathbb{1}$  it follows  $S_3(0.5\mathbb{1}) = \mathbb{1}^2$  and  $S_4(0.5\mathbb{1}) = 2$  (remind that  $\mathbb{1} \cdot \mathbb{1}^{-1} = \mathbb{1}^0 = 1$  (see (11)). It can be seen from this example that it is possible to obtain finite numbers as the result of summing up infinitesimals.

The infinite and infinitesimal numbers allow us to calculate also arithmetic and geometric series with an infinite number of items. For the arithmetic progression,  $a_n = a_1 + (n - 1)d$ , for both finite and infinite  $n$  we have

$$\sum_{i=1}^n a_i = \frac{n}{2}(a_1 + a_n).$$

Then, for instance, the sum of all natural numbers from 1 to  $\mathbb{1}$  can be calculated as follows

$$1 + 2 + 3 + \dots + (\mathbb{1} - 1) + \mathbb{1} = \sum_{i=1}^{\mathbb{1}} i = \frac{\mathbb{1}}{2}(1 + \mathbb{1}) = 0.5\mathbb{1}^2 0.5\mathbb{1}. \quad (12)$$

If we calculate now the following sum of infinitesimals where each item is  $\mathbb{1}$  times less than the corresponding item of (12)

$$\mathbb{1}^{-1} + 2\mathbb{1}^{-1} + \dots + (\mathbb{1} - 1) \cdot \mathbb{1}^{-1} + \mathbb{1} \cdot \mathbb{1}^{-1} = \sum_{i=1}^{\mathbb{1}} i\mathbb{1}^{-1} = \frac{\mathbb{1}}{2}(\mathbb{1}^{-1} + 1) = 0.5\mathbb{1}^1 0.5.$$

then, obviously, the obtained number,  $0.5\mathbb{1}^1 0.5$ , is  $\mathbb{1}$  times less than the sum in (12). This example shows, particularly, that infinite numbers can also be obtained as the result of summing up infinitesimals.



Let us consider now the geometric series  $\sum_{i=0}^{\infty} x^i$ . Traditional analysis proves that it converges to  $\frac{1}{1-x}$  for  $x$  such that  $-1 < x < 1$ . We are able to give a more precise answer for *all* values of  $x$ . To do this, we should fix the number of items in the sum. If we suppose that it contains  $n$  items, where  $n$  is finite or infinite, then

$$Q_n = \sum_{i=0}^n x^i = 1 + x + x^2 + \dots + x^n. \quad (13)$$

By multiplying the left hand and the right hand parts of this equality by  $x$  and by subtracting the result from (13) we obtain

$$Q_n - xQ_n = 1 - x^{n+1}$$

and, as a consequence, for all  $x \neq 1$  the formula

$$Q_n = (1 - x^{n+1})(1 - x)^{-1} \quad (14)$$

holds for finite and infinite  $n$ . Thus, the possibility to express infinite and infinitesimal numbers allows us to take into account infinite  $n$  and the value  $x^{n+1}$  being infinitesimal for a finite  $x$ . Moreover, we can calculate  $Q_n$  for infinite and finite values of  $n$  and  $x = 1$ , because in this case we have just

$$Q_n = \underbrace{1 + 1 + 1 + \dots + 1}_{n+1 \text{ items}} = n + 1.$$

Let us illustrate the obtain results by some examples. In the first of them we consider the divergent series

$$1 + 3 + 9 + \dots = \sum_{i=0}^{\infty} 3^i.$$

In the new language, to define a sum it is necessary to fix the number of items,  $n$ , in it and  $n$  can be finite or infinite. For example, for the infinite  $n = \mathbb{1}^2$  we obtain

$$\sum_{i=0}^{\mathbb{1}^2} 3^i = 1 + 3 + 9 + \dots + 3^{\mathbb{1}^2} = \frac{1 - 3^{\mathbb{1}^2+1}}{1 - 3} = 0.5(3^{\mathbb{1}^2+1} - 1).$$

Analogously, for  $n = \mathbb{1}^2 + 1$  we obtain

$$1 + 3 + 9 + \dots + 3^{\mathbb{1}^2} + 3^{\mathbb{1}^2+1} = 0.5(3^{\mathbb{1}^2+2} - 1).$$

If we now find the difference between the two sums

$$0.5(3^{\mathbb{1}^2+2} - 1) - 0.5(3^{\mathbb{1}^2+1} - 1) = 3^{\mathbb{1}^2+1}(0.5 \cdot 3 - 0.5) = 3^{\mathbb{1}^2+1}$$

we obtain the newly added item  $3^{\mathbb{1}^2+1}$ .

Let us now consider the series  $\sum_{i=1}^{\infty} \frac{1}{2^i}$ . It is well known that it converges to one. However, we are able to give a more precise answer. In fact, the formula

$$\sum_{i=1}^n \frac{1}{2^i} = \frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} \right) = \frac{1}{2} \cdot \frac{1 - \frac{1}{2}^n}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

can be used directly for infinite  $n$ , too. For example, if  $n = \textcircled{1}$  then

$$\sum_{i=1}^{\textcircled{1}} \frac{1}{2^i} = 1 - \frac{1}{2^{\textcircled{1}}},$$

where  $\frac{1}{2^{\textcircled{1}}}$  is infinitesimal. Thus, the traditional answer  $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$  was just a finite approximation to our more precise result using infinitesimals.

In order to show the potential of the new approach for the work with divergent series with alternate signs, we start from the famous series

$$S_5 = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

In literature, there exist many approaches giving different answers regarding the value of this series (see [13]). All of them use various notions of average to calculate the series. However, the notions of the sum and an average are different. In our approach, we do not appeal to an average and calculate the required sum directly. To do this we should indicate explicitly the number of items,  $k$ , in the sum. Then

$$S_5(k) = \underbrace{1 - 1 + 1 - 1 + 1 - 1 + \dots}_{k \text{ items}} = \begin{cases} 0, & \text{if } k = 2n, \\ 1, & \text{if } k = 2n + 1, \end{cases} \quad (15)$$

where  $k$  can be finite or infinite. For example,  $S_5(\textcircled{1}) = 0$  because  $\textcircled{1}$  is even (see [17]). Analogously,  $S_5(\textcircled{1} - 1) = S_5(2\textcircled{1} + 1) = 1$  because both  $\textcircled{1} - 1$  and  $2\textcircled{1} + 1$  are odd.

It is important to emphasize that the answers obtained in all the examples considered above (including the latter related to the divergent series  $S_5(k)$  with alternate signs) do not depend on the way the items in the entire sum are re-arranged. In fact, since we always know the exact infinite number of items in the sum and the order of alternating the signs is clearly defined, we know also the exact number of positive and negative items in the sum.

Suppose, for instance, that we want to re-arrange the items in the sum  $S_5(\textcircled{1})$  in the following way

$$1 + 1 - 1 + 1 + 1 - 1 + 1 + 1 - 1 + \dots$$

However, we know that the sum  $S_5(\textcircled{1})$  has  $\textcircled{1}$  items and that grossone is even. This means that in the sum there are  $\frac{\textcircled{1}}{2}$  positive and  $\frac{\textcircled{1}}{2}$  negative items. As a result, the re-arrangement considered above can continue only until the positive items will

not finish and then it will be necessary to continue to add only negative numbers. More precisely, we have

$$S_5(\textcircled{1}) = \underbrace{1 + 1 - 1 + 1 + 1 - 1 + \dots + 1 + 1 - 1}_{\frac{\textcircled{1}}{2} \text{ positive and } \frac{\textcircled{1}}{4} \text{ negative items}} \underbrace{- 1 - 1 - \dots - 1 - 1 - 1}_{\frac{\textcircled{1}}{4} \text{ negative items}} = 0, \quad (16)$$

where the result of the first part in this re-arrangement is calculated as  $(1 + 1 - 1) \cdot \frac{\textcircled{1}}{4} = \frac{\textcircled{1}}{4}$  and the result of the second part is equal to  $-\frac{\textcircled{1}}{4}$ .

Note that the record (16) is a re-arrangement of the sum introduced in (15) where the order of the alternating of positive and negative items has been defined. Thanks to this order and to the knowledge of the whole number of items in the sum (15) we were able to calculate the number of negative and positive items. Obviously, if we consider another sum where the order of the alternating of the signs and the number of the items are given in a different way then the result can be also different. For instance, if we have the sum

$$S_6(\textcircled{1}) = \underbrace{1 + 1 - 1 + 1 + 1 - 1 + \dots}_{\textcircled{1} \text{ items}}$$

then we have  $\frac{2\textcircled{1}}{3}$  positive items and  $\frac{\textcircled{1}}{3}$  negative items that gives the result  $S_6(\textcircled{1}) = \frac{\textcircled{1}}{3}$ .

Let us consider now the following divergent series

$$S_7 = 1 - 2 + 3 - 4 + \dots$$

The corresponding sum  $S_7(k)$  can be easily calculated as the difference of two arithmetic progressions after we have fixed the infinite number of items,  $k$ , in it. Suppose that  $k = \textcircled{1}$ . Then it follows

$$\begin{aligned} S_7(\textcircled{1}) &= \underbrace{1 - 2 + 3 - 4 + \dots - (\textcircled{1} - 2) + (\textcircled{1} - 1) - \textcircled{1}}_{\textcircled{1} \text{ items}} = \\ &= \underbrace{(1 + 3 + 5 + \dots + (\textcircled{1} - 3) + (\textcircled{1} - 1))}_{\frac{\textcircled{1}}{2} \text{ items}} - \underbrace{(2 + 4 + 6 + \dots + (\textcircled{1} - 2) + \textcircled{1})}_{\frac{\textcircled{1}}{2} \text{ items}} = \\ &= \frac{(1 + (\textcircled{1} - 1))\textcircled{1}}{4} - \frac{(2 + \textcircled{1})\textcircled{1}}{4} = \frac{\textcircled{1}^2 - 2\textcircled{1} - \textcircled{1}^2}{4} = -\frac{\textcircled{1}}{2}. \end{aligned} \quad (17)$$

Obviously, if we change the number of items,  $k$ , then, as it happens in the finite case, the results of summation will also change. For instance, it follows

$$S_7(\textcircled{1} - 1) = \frac{\textcircled{1}}{2}, \quad S_7(\textcircled{1} + 1) = \frac{\textcircled{1}}{2} + 1, \quad S_7(\textcircled{1}^2) = -\frac{\textcircled{1}^2}{2}.$$

In particular, for  $k = \textcircled{1} - 1$  we have

$$S_7(\textcircled{1} - 1) = \underbrace{1 - 2 + 3 - 4 + \dots - (\textcircled{1} - 2) + (\textcircled{1} - 1)}_{\textcircled{1}-1 \text{ items}}$$

$$\begin{aligned} & \underbrace{(1+3+5+\dots+(\mathbb{1}-3)+(\mathbb{1}-1))}_{\frac{\mathbb{1}}{2} \text{ items}} - \underbrace{(2+4+6+\dots+(\mathbb{1}-2))}_{\frac{\mathbb{1}}{2}-1 \text{ items}} = \\ & \frac{(1+(\mathbb{1}-1))\mathbb{1}}{4} - \frac{(2+(\mathbb{1}-2))(\mathbb{1}/2-1)}{2} = \frac{\mathbb{1}^2}{4} - \left( \frac{\mathbb{1}^2}{4} - \frac{\mathbb{1}}{2} \right) = \frac{\mathbb{1}}{2}. \end{aligned}$$

Obviously, we have (cf. (17)) that

$$S_7(\mathbb{1}) = S_7(\mathbb{1}-1) - \mathbb{1} = \frac{\mathbb{1}}{2} - \mathbb{1} = -\frac{\mathbb{1}}{2}.$$

Analogously to the passage from series to sums, the introduction of infinite and infinitesimal numerals allows us (in fact, it *imposes*) to substitute improper integrals of various kinds by integrals defined in a more precise way. For example, let us consider the following improper integral

$$\int_0^\infty x^2 dx.$$

Now it is necessary to define its upper infinite limit of integration explicitly. Then, different infinite numbers used instead of  $\infty$  will lead to different results, as it happens in the finite case. For instance, numbers  $\mathbb{1}$  and  $\mathbb{1}^2$  give us two different integrals both assuming infinite but different values:

$$\int_0^{\mathbb{1}} x^2 dx = \frac{1}{3}\mathbb{1}^3, \quad \int_0^{\mathbb{1}^2} x^2 dx = \frac{1}{3}\mathbb{1}^6.$$

Moreover, it becomes possible to calculate integrals where both endpoints of the interval of integration are infinite as in the following example

$$\int_{\mathbb{1}}^{\mathbb{1}^2} x^2 dx = \frac{1}{3}\mathbb{1}^6 - \frac{1}{3}\mathbb{1}^3.$$

We conclude this section by emphasizing that, as it was with sums, it becomes also possible to calculate integrals of functions assuming infinite and infinitesimal values and the obtained results of integration can be finite, infinite, and infinitesimal. For instance, in the integral

$$\int_{\mathbb{1}}^{\mathbb{1}+\mathbb{1}^{-2}} x^2 dx = \frac{1}{3}(\mathbb{1}^1 + \mathbb{1}^{-2})^3 - \frac{1}{3}\mathbb{1}^3 = 1\mathbb{1}^0 + 1\mathbb{1}^{-3} + \frac{1}{3}\mathbb{1}^{-6}$$

the result has a finite part and two infinitesimal parts and in the integral

$$\int_{\mathbb{1}}^{\mathbb{1}+\mathbb{1}^{-2}} x^2 - x dx = 1\mathbb{1}^{-3} - \frac{1}{2}\mathbb{1}^{-4} + \frac{1}{3}\mathbb{1}^{-6}$$

the result has three infinitesimal parts. The last two examples illustrate situations when the integrand is infinite

$$\int_{\textcircled{1}}^{\textcircled{1}^2} \textcircled{1} x^2 dx = \textcircled{1} \int_{\textcircled{1}}^{\textcircled{1}^2} x^2 dx = \frac{1}{3} \textcircled{1}^7 - \frac{1}{3} \textcircled{1}^4,$$

and infinitesimal

$$\int_{\textcircled{1}}^{\textcircled{1}^2} \textcircled{1}^{-4} x^2 dx = \textcircled{1}^{-4} \int_{\textcircled{1}}^{\textcircled{1}^2} x^2 dx = \frac{1}{3} \textcircled{1}^2 - \frac{1}{3} \textcircled{1}^{-1}.$$

#### 4 Methodological consequences and analysis of some well known results related to the zeta function

Let us return now to the Riemann zeta function and consider it together with some Euler's results from the new methodological positions. The first remark we can make is that the records of the type

$$\begin{aligned} f_1(x) &= \sum_{i=1}^{\infty} a_i(x), & f_2(x) &= \sum_{-\infty}^{\infty} a_i(x), \\ f_3(x) &= \int_a^{\infty} g(t,x) dt, & f_4(x) &= \int_{-\infty}^{\infty} g(t,x) dt, \end{aligned} \tag{18}$$

and similar are sufficiently precise to define a function only if one uses the traditional language. As it has been shown in the previous section, the accuracy of the answers one can get with this language is lower with respect to the new numeral system using  $\textcircled{1}$ .

Thanks to this numeral system, we know now that the only symbol  $\infty$  and the absence of numerals allowing us to express different infinite numbers just did not give us the possibility to distinguish many different functions in each of the objects present in (18). We can say that we are now in the situation similar to a physicist who first observed a black dot in a weak lens and then had put a stronger one and has seen that that the black dot consists of many different dots. When we have changed the lens in our mathematical microscope we have seen that within each object of our study viewed as one entity with the old lens we can distinguish many different separate objects. This means that we cannot use the traditional language when the accuracy of the answers to questions made with respect to (18) is expected to be as high as possible.

Now, when we are aware both of the existence of different infinite numbers and of the numeral system of Pirahã, records like (18) can be written also as

$$\begin{aligned} f_1(x) &= \sum_{i=1}^{\text{many}} a_i(x), & f_2(x) &= \sum_{-\text{many}}^{\text{many}} a_i(x), \\ f_3(x) &= \int_a^{\text{many}} g(t,x) dt, & f_4(x) &= \int_{-\text{many}}^{\text{many}} g(t,x) dt. \end{aligned}$$

We emphasize again that records like (18) are not wrong, they are just less precise than records that could be written using various infinite numerals. In the moment

when a mathematician has decided which numeral system he/she will use in his/her work, he/she has decided which mathematical objects he/she will be able to observe and to describe. Thus, if the accuracy of the records (18) is sufficient for the problem under consideration, than everything is fine. Is this the case of the problem of zeros of the Riemann zeta function? Let us see.

The fact that within  $\infty$  many different infinite numbers can be distinguished means that records (1) and (5) do not describe two single functions and there exist *many different* Riemann zeta functions (and, obviously, many different Dirichlet eta functions) and to fix a concrete one it is necessary to fix the number of items in the corresponding sum. In order to define a Riemann zeta function (a Dirichlet eta function), we should choose an infinite number  $n$  expressible in a numeral system using grossone (for instance, we can take the numeral system briefly described in Section 2) and then write

$$\zeta(s, n) = \sum_{u=1}^n \frac{1}{u^s}, \quad \eta(s, n) = \sum_{u=1}^n \frac{(-1)^{u-1}}{u^s}. \quad (19)$$

As a result, for different (infinite or finite) values of  $n$  we have different functions. For example, for  $n = \mathbb{1}/2$  and  $n = \mathbb{1}$  we get

$$\zeta(s, \mathbb{1}/2) = \sum_{u=1}^{\mathbb{1}/2} \frac{1}{u^s}, \quad \zeta(s, \mathbb{1}) = \sum_{u=1}^{\mathbb{1}} \frac{1}{u^s}$$

that are two different functions and, analogously,

$$\eta(s, \mathbb{1}) = \sum_{u=1}^{\mathbb{1}} \frac{(-1)^{u-1}}{u^s}, \quad \eta(s, \mathbb{1}/2) = \sum_{u=1}^{\mathbb{1}/2} \frac{(-1)^{u-1}}{u^s},$$

are also two different functions.

Note that there obviously exist questions for which both mathematical languages used in (1) and (19) are sufficiently accurate. This can happen when the property we ask about holds for all (finite and infinite) values of  $n$  in (19). For instance, it is possible easily to answer in the affirmative to the question: ‘Is the inequality  $\zeta(1) > -100$  correct?’ because this result holds for all values of  $n$  in (19).

The above observations mean that it becomes necessary to reconsider classical results concerning properties of the Riemann zeta function (1) in order to re-write them (where it is possible) for the form (19) with infinite values of  $n$ . First, it is easy to understand that the relation (4) can be re-written as

$$\eta(s, n) = \zeta(s, 2k) - 2^{1-s} \zeta(s, k) \quad (20)$$

for even values of  $n = 2k$  and as

$$\eta(s, n) = \zeta(s, 2k+1) - 2^{1-s} \zeta(s, k) \quad (21)$$

for  $n = 2k+1$ .

Let us look now at the formula (2). It has been first introduced and studied by Euler (see [8, 9, 10, 11]) who proved for integer values of  $s$  the famous identity

$$\sum_{u=1}^{\infty} \frac{1}{u^s} = \prod_{p \text{ primes}} \frac{1}{1-p^{-s}} \quad (22)$$

being one of the strongest sources of the interest to the Riemann zeta function. This identity has been proved using the traditional language and, as a consequence, its accuracy depends on the accuracy of the used language, i.e., the language that cannot distinguish within  $\infty$  different infinite numbers.

If one wants to use the new language and to have a higher accuracy then it is not possible to write with nonchalance identities involving infinite sums and/or products of the type

$$\sum_{u=1}^{\infty} a_i = \prod_{i=1}^{\infty} b_i.$$

It is necessary to take care on the number of infinite items both in the sum and in the product in the same way as we do it for the finite number of items. These identities should be substituted, where this is possible, by the identities

$$\sum_{u=1}^n a_i = \prod_{i=1}^k b_i,$$

where both  $n$  and  $k$  are infinite (and probably different) numbers. Proofs of such identities should be reconsidered with the accuracy imposed by the new language.

It can easily happen that a result being correct with the accuracy of one language is not sufficiently precise when a language with a higher accuracy is used. We recall that for Pirahã the records  $2 + 1 =$  ‘many’ and  $2 + 2 =$  ‘many’ are correct. This means that in their language  $2 + 1 = 2 + 2 =$  ‘many’. Note that also for us, people knowing that  $2 + 1 = 3$  and  $2 + 2 = 4$ , the result of Pirahã is not wrong, it is just *inaccurate*. Their mathematical language cannot express the numbers ‘three’ and ‘four’ but for their purposes this low accuracy of answers is sufficient.

Let us consider the Euler product formula (22) and compare in this occasion the traditional language working with  $\infty$  with the new one using ①. To prove (22) Euler expands each of the factors of the right hand part of (22) as follows

$$\frac{1}{1 - \frac{1}{p^s}} = 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots \quad (23)$$

Then he observes that their product is therefore a sum of terms of the form

$$\frac{1}{(p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_r^{n_r})^s}, \quad (24)$$

where  $p_1, p_2, \dots, p_r$  are distinct primes and  $n_1, n_2, \dots, n_r$  are natural numbers. Euler then uses the fundamental theorem of arithmetic (every integer can be written as a

product of primes) and concludes that the sum of all items (24) is the left hand part of (22).

In the new language this way of doing is not acceptable because, in order to start, we should indicate the precise infinite number,  $n$ , of items in the sum in the left hand part of (22). This operation gives us different functions  $\zeta(s, n)$  from (19). Then (23) should be rewritten by indicating the exact infinite number,  $k$ , of items in its right hand part and the result of summing up will be different with respect to (22). Namely, we have

$$1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots + \frac{1}{(p^{k-1})^s} = \frac{1 - \frac{1}{(p^k)^s}}{1 - \frac{1}{p^s}}. \quad (25)$$

The accuracy of the language used by Euler did not allow him either to observe different infinite values of  $n$  and  $k$  or to take into account the infinitesimal value  $\frac{1}{(p^k)^s}$ .

It is necessary to emphasize that the analysis made above does not mean that (22) is not correct. Euler uses the language involving the symbol  $\infty$  and the accuracy of this language does not allow him (and actually anyone) to perform a more precise analysis. Analogously, Pirahã have  $2 + 1 =$  ‘many’ and  $2 + 2 =$  ‘many’ and these results are correct in their language. Now we have numerals 3 and 4 and are able to obtain more accurate answers and to observe that  $2 + 1 = 3 \neq 4 = 2 + 2$ . Both results,  $2 + 1 =$  ‘many’  $= 2 + 2$  and  $2 + 1 = 3 \neq 4 = 2 + 2$ , are correct but with different accuracies determined by the languages used for calculations. Both languages can be used in dependence on the problem one wishes to deal with.

Another metaphor that can help is the following. Suppose that we have measured two distances  $A$  and  $B$  with the accuracy equal to 1 meter and we have found that both of them are equal to 25 meters. Suppose now that we want to measure them with the accuracy equal to 1 centimeter. Then, very probably, we shall obtain something like  $A = 2487$  centimeters and  $B = 2538$  centimeters, i.e.,  $A \neq B$ . Both answers,  $A = B$  and  $A \neq B$ , are correct but with different accuracies and both of them can be used successfully in different situations. For instance, if one just wants to go for a walk, then the accuracy of the answer  $A = B$  expressed in meters is sufficient. However, if one needs to connect some devices with a cable, then a higher accuracy is required and the answer expressed in centimeters should be used.

We are with the Euler product formula in the same situation. It is correct with the accuracy of the language using  $\infty$  because this language does not allow one to distinguish different infinite numbers within  $\infty$ . In the same time, when a language allows us to distinguish different infinite and infinitesimal numbers, it follows from (25) that for any infinite (or finite)  $k$  we have

$$1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots + \frac{1}{(p^{k-1})^s} \neq \frac{1}{1 - p^{-s}} \quad (26)$$

and, as a result, the following theorem holds.



**Theorem 4.1.** For prime  $p_i$  and both finite and infinite values of  $n$  and  $k$  it follows that

$$\sum_{u=1}^n \frac{1}{u^s} \neq \prod_{i=1}^k \frac{1}{1-p_i^{-s}}. \quad (27)$$

Thus, the choice of the language fixes the accuracy of the analysis one can perform. In this paper, we show that the traditional language using only the symbol  $\infty$  is not sufficiently precise when one wishes to work with the Riemann zeta function and the Dirichlet eta function.

Let us comment upon another famous result of Euler related to the Riemann zeta function – his solution to the Basel problem where he has shown that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}. \quad (28)$$

Let us first briefly present Euler's proof and then comment upon it. Note that from the point of view of modern mathematics this proof can be criticized from different points of view. Nowadays there exist many other proofs considered by mathematicians as more accurate<sup>2</sup>. However, since also in the modern proofs the symbol  $\infty$  and the concept of series are used, the analysis made below can be applied to the other proofs, as well.

Euler begins with the standard Taylor expansion of  $\sin(x)$ ,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \quad (29)$$

which converges for all  $x$ . Euler interprets the left hand side of (29) as an "infinite polynomial"  $P(x)$  that, therefore, can be written as product of factors based on its roots. Since the roots of  $\sin(x)$  are  $\dots - 3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \dots$  then it follows

$$P(x) = Cx(x^2 - \pi^2)(x^2 - 4\pi^2)(x^2 - 9\pi^2)\dots$$

Euler continues by reminding that  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$  and

$$\lim_{x \rightarrow 0} \frac{x(x^2 - \pi^2)(x^2 - 4\pi^2)(x^2 - 9\pi^2)\dots}{x} = (-\pi^2)(-4\pi^2)(-9\pi^2)\dots$$

Thus,  $C$  should be the reciprocal of the infinite product on the right and we obtain

$$\sin(x) = x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \dots \quad (30)$$

Then, as follows from (29) and (30), Euler has written  $P(x)$  in two different ways and he equates these two records

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \dots, \quad (31)$$

<sup>2</sup>This fact is another manifestation of the continuous mutation of mathematical languages. The views on accuracy of proofs have changed since Euler's times and the modern language is considered more accurate.

Now Euler equates the coefficients of  $x^3$  on both sides of (30) and gets first

$$-\frac{1}{3} = -\frac{1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \frac{1}{4^2\pi^2} \cdots,$$

and then the final beautiful result (28).

Let us now consider these results using the new approach. We take the function  $\sin(x)$  introduced using the standard trigonometric reasoning. Note that such a definition just describes its properties and does not tell us how to *calculate*  $\sin(x)$  precisely at all  $x$ . When one uses the trigonometric definition, values of  $\sin(x)$  only at certain  $x$  are known precisely, e.g.,  $\sin(\frac{\pi}{2}) = 1$ ,  $\sin(2\pi) = 0$ , etc. Notice that in these records we do not use any approximation of the number  $\pi$ . Similarly to  $\sin(x)$ , it is described by its properties and a special numeral,  $\pi$ , is introduced to indicate it. Then, those values of  $\sin(x)$  that are not linked to geometric ideas are defined through various approximations of both  $\sin(x)$  and  $\pi$ .

The new language taken together with the trigonometric definition of  $\sin(x)$  allows us to evaluate  $\sin(x)$  precisely not only at certain finite points but also at certain infinite points. For instance, it follows  $\sin(2\mathbb{Q}\pi) = 0$ ,  $\sin(\mathbb{Q}\pi + \frac{\pi}{2}) = 1$ , etc.

Then, the explicit usage of infinite and infinitesimal numbers imposes us to move from (29) to the approximation with the polynomial  $P_1(x, 2k+1)$  of the order  $2k+1$  where

$$\sin(x) \approx P_1(x, 2k+1) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad k = 0, 1, 2, \dots \quad (32)$$

and for different finite or infinite  $k$  we get different approximations. Analogously, the idea used in (30) gives us the second kind of approximation where the polynomial  $P_2(x, 2n+1)$  of the order  $2n+1$  is used

$$\begin{aligned} \sin(x) &\approx P_2(x, 1) = x, \\ \sin(x) &\approx P_2(x, 2n+1) = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2}) \dots (1 - \frac{x^2}{n^2\pi^2}), \quad n = 1, 2, \dots \end{aligned} \quad (33)$$

where for different finite or infinite values of  $n$  we get different approximations.

Obviously, when  $k \neq n$ , it follows  $P_1(x, 2k+1) \neq P_2(x, 2n+1)$ . In the case  $k = n$ , two polynomials will be equal if all their coefficients are equal. Following Euler we first equate the coefficients of  $x^3$  and obtain

$$-\frac{1}{3} = -\frac{1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \frac{1}{4^2\pi^2} \cdots - \frac{1}{k^2\pi^2},$$

From where we get

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{k^2}. \quad (34)$$

However, such a kind of equating should be done for *all the coefficients* of  $P_1(x, 2k+1)$  and  $P_2(x, 2n+1)$ . Suppose that  $k$  in the sum (32) is even, by equating the coefficients of the highest power  $2k+1$  of  $x$  in two polynomials we get that it should

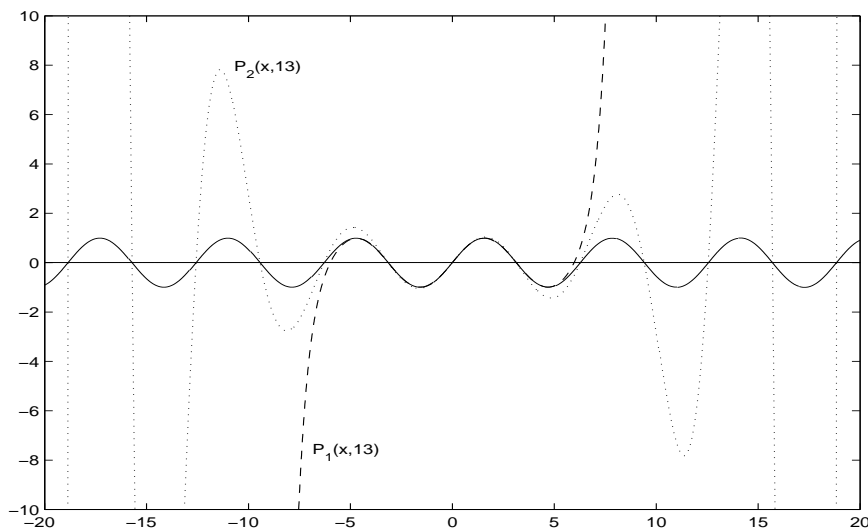


Figure 1: Polynomials  $P_1(x, 13)$  and  $P_2(x, 13)$  together with  $\sin(x)$ .

be

$$\frac{1}{(2k+1)!} = \frac{1}{(k!)^2 \pi^{2k}},$$

from where we have

$$\frac{\pi^{2k}}{(2k+1)!} = \frac{1}{(k!)^2}. \quad (35)$$

Equating coefficients for other powers of  $x$  will give us more different expressions to be satisfied and it is easy to see that they cannot hold *all together*.

Thus, polynomials  $P_1(x, 2k+1)$  and  $P_2(x, 2n+1)$  give us different approximations of  $\sin(x)$ . As an example we show in Fig. 1 polynomials  $P_1(x, 13)$  and  $P_2(x, 13)$  together with  $\sin(x)$ . Higher is the order of the polynomials better are the approximations but the differences are always present for both finite and infinite values of  $k$  and  $n$ .

As it was with the product formula, Euler results are correct with the accuracy of the traditional mathematical language working only with the symbol  $\infty$ . It does not allow one to distinguish different infinite values of  $k$  and  $n$  and, as a consequence, different infinite polynomials  $P_1(x, 2k+1)$  and  $P_2(x, 2n+1)$ .

Since all the traditional computations are executed with finite values of  $k$  and  $n$  and only the initial finite part of (35) is used, it is always possible to choose sufficiently high finite values of  $k$  and  $n$  giving so valid approximations of  $\pi$  with a *finite* accuracy. We are not able by using the traditional mathematical tools to observe the behavior of the polynomials and  $\sin(x)$  at various points  $x$  at infinity. Thus, the fact of the difference of these objects in infinity cannot be noticed by finite numeral systems extended only by the symbol  $\infty$ . However, it is perfectly visible when one uses the new numeral system where different infinite and infinitesimal

numbers can be expressed and distinguished. Thus, we can conclude this section by the following theorem that has been just proved.

**Theorem 4.2.** *For both finite and infinite values of  $n$  and  $k$  it follows that*

$$\sin(x) \neq P_1(x, 2k+1) \neq P_2(x, 2n+1). \quad (36)$$

## 5 Calculating $\zeta(s, n)$ and $\eta(s, n)$ for infinite values of $n$ at $s = 0$ and finite points $s = -1, -2, -3, \dots$

In this section, we establish some results for functions  $\zeta(s, n)$  and  $\eta(s, n)$  from (19) using the approach presented in Section 3. The analysis made there allows us to calculate  $\zeta(s, n)$  and  $\eta(s, n)$  for infinite values of  $n$  at  $s = 0$  and  $s = -1$ :

$$\zeta(0, n) = \underbrace{1 + 1 + 1 + \dots + 1}_n = n,$$

$$\zeta(-1, n) = 1 + 2 + 3 + \dots + n = \frac{(n+1)n}{2}.$$

From where, for instance, at  $n = \mathbb{1}/2$  we obtain

$$\begin{aligned} \zeta(0, \mathbb{1}/2) &= \mathbb{1}/2, & \zeta(0, \mathbb{1}) &= \mathbb{1}, \\ \zeta(-1, \mathbb{1}/2) &= \frac{(\mathbb{1}/2 + 1)\mathbb{1}}{4} & \zeta(-1, \mathbb{1}) &= \frac{(\mathbb{1} + 1)\mathbb{1}}{2}. \end{aligned}$$

Analogously, calculation of the function,  $\eta(s, n)$ , at the points  $s = 0$  and  $s = -1$  for infinite values of  $n$  also has been discussed above. We obtain

$$\eta(0, n) = \underbrace{1 - 1 + 1 - 1 + 1 - 1 + 1 - \dots}_n = \begin{cases} 0, & \text{if } n = 2k, \\ 1, & \text{if } n = 2k + 1, \end{cases}$$

for example,  $\eta(0, \mathbb{1}) = 0$  and  $\eta(0, \mathbb{1}^2 - 1) = 1$ . Analogously, in (17) we have seen how it is possible to calculate  $\eta(-1, n)$  for infinite values of  $n$ . For instance, it follows  $\eta(-1, \mathbb{1}) = -\frac{\mathbb{1}}{2}$ . Note that these results fit well relation (20). For example, for  $n = \mathbb{1}$  it follows

$$\zeta(0, \mathbb{1}) - 2\zeta(0, \mathbb{1}/2) = 0 = \eta(0, \mathbb{1}),$$

$$\zeta(-1, \mathbb{1}) - 4\zeta(-1, \mathbb{1}/2) = -\frac{\mathbb{1}}{2} = \eta(-1, \mathbb{1}).$$

Let us present now a general method for calculating the values of  $\zeta(s, n)$  and  $\eta(s, n)$  for infinite or finite  $n$  at  $s$  being negative finite integers. For this purpose we use the idea of Euler to multiply the identity

$$f(x) = \sum_{i=0}^n x^i = 1 + x + x^2 + \dots + x^n, \quad (37)$$

by  $x$ . By using the differential calculus developed in [19] we can apply this idea without distinction for both finite and infinite values of  $n$  and  $x \neq 1$  that can be finite, infinite or infinitesimal. By differentiation of (37) we obtain

$$f'(x) = \sum_{i=0}^{n-1} ix^{i-1} = 1 + 2x + 3x^2 + \dots + nx^{n-1}, \quad (38)$$

Since we have for finite, infinite or infinitesimal  $x \neq 1$  that

$$f(x) = \frac{1 - x^{n+1}}{1 - x} \quad (39)$$

then by differentiation of (39) we have that

$$f'(x) = \frac{1 + nx^{n+1} - (n+1)x^n}{(1-x)^2}. \quad (40)$$

Thus, we can conclude that for finite and infinite  $n$  and for finite, infinite or infinitesimal  $x \neq 1$  it follows

$$\frac{1 + nx^{n+1} - (n+1)x^n}{(1-x)^2} = \sum_{i=0}^{n-1} ix^{i-1} = 1 + 2x + 3x^2 + \dots + nx^{n-1}. \quad (41)$$

By taking  $x = -1$  we obtain that the value of  $\eta(-1, n)$  is

$$\eta(-1, n) = 1 - 2 + 3 - 4 + \dots + n(-1)^{n-1} = \frac{1 + n(-1)^{n+1} - (n+1)(-1)^n}{4}.$$

For example, for  $n = \mathbb{1}$  we obtain the result that has been already calculated in (17) in a different way

$$\begin{aligned} \eta(-1, \mathbb{1}) &= 1 - 2 + 3 - 4 + \dots + \mathbb{1}(-1)^{\mathbb{0}-1} = \\ &= \frac{1 + \mathbb{1}(-1)^{\mathbb{1}+1} - (\mathbb{1}+1)(-1)^{\mathbb{1}}}{4} = \frac{1 - \mathbb{1} - (\mathbb{1}+1)}{4} = -\frac{\mathbb{1}}{2}. \end{aligned}$$

If we multiply successively (41) by  $x$  and use again differentiation of both parts of the obtained equalities then it becomes possible to obtain the values of  $\eta(s, n)$  for other finite integer negative points  $s$ . Thus, in order to obtain, for instance, formulae for  $s = -2, -3, -4$ , we proceed as follows

$$1 + 2^2x + 3^2x^2 + \dots + n^2x^{n-1} = \frac{-n^2x^{n+2} + (2n^2 + 2n - 1)x^{n+1} - (n+1)^2x^n + x + 1}{(1-x)^3}, \quad (42)$$

$$\begin{aligned} 1 + 2^3x + 3^3x^2 + 4^3x^3 \dots + n^3x^{n-1} &= (n^3x^{n+3} - (3n^3 + 3n^2 - 3n + 1)x^{n+2} + \\ &+ (3n^3 + 6n^2 - 4)x^{n+1} - (n+1)^3x^n + x^2 + 4x + 1)(1-x)^{-4}, \quad (43) \end{aligned}$$

$$1 + 2^4x + 3^4x^2 + 4^4x^3 \dots + n^4x^{n-1} = (n^4x^{n+4} - (4n^4 + 4n^3 - 6n^2 + 4n - 1)x^{n+3} +$$

$$(6n^4 + 12n^3 - 6n^2 - 12n + 11)x^{n+2} - (4n^4 + 12n^3 + 6n^2 - 12n - 11)x^{n+1} + (n+1)^4x^n - x^3 - 11x^2 - 11x - 1)(x-1)^{-5}. \quad (44)$$

By taking  $x = -1$  in (42), (43), and (44) we obtain that

$$\eta(-2, n) = 2^{-3}(-n^2(-1)^{n+2} + (2n^2 + 2n - 1)(-1)^{n+1} - (n+1)^2(-1)^n),$$

$$\eta(-3, n) = 2^{-4}(n^3(-1)^{n+3} - (3n^3 + 3n^2 - 3n + 1)(-1)^{n+2} + (3n^3 + 6n^2 - 4)(-1)^{n+1} - (n+1)^3(-1)^n - 2),$$

$$\eta(-4, n) = -2^{-5}(n^4(-1)^{n+4} - (4n^4 + 4n^3 - 6n^2 + 4n - 1)(-1)^{n+3} + (6n^4 + 12n^3 - 6n^2 - 12n + 11)(-1)^{n+2} - (4n^4 + 12n^3 + 6n^2 - 12n - 11)(-1)^{n+1} + (n+1)^4(-1)^n).$$

Then, by taking different infinite values of  $n$  we are able to calculate the respective values of the functions. For example, for  $n = \mathbb{1}$  it follows

$$\eta(-2, \mathbb{1}) = 2^{-3}(-\mathbb{1}^2 - 2\mathbb{1}^2 - 2\mathbb{1} + 1 - (\mathbb{1} + 1)^2) = -0.5\mathbb{1}(\mathbb{1} + 1), \quad (45)$$

$$\eta(-3, \mathbb{1}) = 2^{-4}(-\mathbb{1}^3 - (3\mathbb{1}^3 + 3\mathbb{1}^2 - 3\mathbb{1} + 1) - (3\mathbb{1}^3 + 6\mathbb{1}^2 - 4) - (\mathbb{1} + 1)^3 - 2) = -0.5\mathbb{1}^2(\mathbb{1} + 3), \quad (46)$$

$$\eta(-4, \mathbb{1}) = -2^{-5}(\mathbb{1}^4 + 4\mathbb{1}^4 + 4\mathbb{1}^3 - 6\mathbb{1}^2 + 4\mathbb{1} - 1 + 6\mathbb{1}^4 + 12\mathbb{1}^3 - 6\mathbb{1}^2 - 12\mathbb{1} + 11 + 4\mathbb{1}^4 + 12\mathbb{1}^3 + 6\mathbb{1}^2 - 12\mathbb{1} - 11 + (\mathbb{1} + 1)^4) = -0.5\mathbb{1}(\mathbb{1} + 1)(\mathbb{1}^2 + \mathbb{1} - 1). \quad (47)$$

In order to use the same technique for calculating  $\zeta(s, n)$  for infinite or finite  $n$  at negative finite integers  $s < -1$  it would be necessary to evaluate (42), (43), and (44) at the point  $x = 1$ . In traditional mathematics this is impossible whereas the new approach (see [18, 19]) allows us to execute the required evaluations by using the following method.

If we put  $x = 1$  at the left-hand parts of (42), (43), and (44) then we see that we have there infinite sums of positive integers. Thus, these sums should be equal to some infinite positive integers. Since in the right-hand parts of these equalities it is not possible to use  $x = 1$ , we introduce an infinitesimal perturbation,  $\mathbb{1}^{-\alpha}$ ,  $\alpha > 0$ , and calculate the the right-hand parts at the point  $x = 1 + \mathbb{1}^{-\alpha}$  that is infinitesimally close to  $x = 1$ . Then, in the obtained result, we separate the contribution of the perturbation that can be kept infinitesimal by the choice of  $\alpha$ , from the contribution of the point  $x = 1$  that should be equal, as we have established, to an infinite integer.

In order to present the method, let us execute calculations for  $s = -2$ , results for other values of  $s$  are obtained by a complete analogy. We indicate the right-hand part of (42) as  $f(x, n)$  and, by using the usual notation,  $\binom{n}{k}$ , for binomial coefficients, proceed as follows

$$f(1 + \mathbb{1}^{-\alpha}, n) = -\mathbb{1}^{3\alpha} \left( -n^2 \left[ 1 + (n+2)\mathbb{1}^{-\alpha} + \frac{1}{2}(n+1)(n+2)\mathbb{1}^{-2\alpha} + \right. \right.$$

$$\begin{aligned}
& \frac{1}{6}n(n+1)(n+2)\mathbb{1}^{-3\alpha} + \binom{n+2}{4}\mathbb{1}^{-4\alpha} + \dots + \mathbb{1}^{-(n+2)\alpha} \Big] + \\
(2n^2 + 2n - 1) & \left[ 1 + (n+1)\mathbb{1}^{-\alpha} + \frac{1}{2}n(n+1)\mathbb{1}^{-2\alpha} + \frac{1}{6}(n-1)n(n+1)\mathbb{1}^{-3\alpha} + \right. \\
& \left. \binom{n+1}{4}\mathbb{1}^{-4\alpha} + \dots + \mathbb{1}^{-(n+1)\alpha} \right] - (n+1)^2 \left[ 1 + n\mathbb{1}^{-\alpha} + \frac{1}{2}(n-1)n\mathbb{1}^{-2\alpha} + \right. \\
& \left. \frac{1}{6}(n-2)(n-1)n\mathbb{1}^{-3\alpha} + \binom{n}{4}\mathbb{1}^{-4\alpha} + \dots + \mathbb{1}^{-n\alpha} \right] + \mathbb{1}^{-\alpha} + 2).
\end{aligned}$$

By collecting the terms of grossone we then obtain

$$f(1 + \mathbb{1}^{-\alpha}, n) = \frac{1}{6}n(n+1)(2n+1) +$$

$$\left( n^2 \binom{n+2}{4} - (2n^2 + 2n - 1) \binom{n+1}{4} + (n+1)^2 \binom{n}{4} \right) \mathbb{1}^{-\alpha} + \dots + n^2 \mathbb{1}^{-(n+2)\alpha}. \quad (48)$$

As it can be seen from (48), for any finite or infinite value of  $n$  there always can be chosen a number  $\alpha > 0$  such that the contribution of the added infinitesimal  $\mathbb{1}^{-\alpha}$  in  $f(1 + \mathbb{1}^{-\alpha}, n)$  is a sum of infinitesimals (see the second line of (48)). Due to the representation (9), (10), this contribution can be easily separated from the integer finite or infinite part represented by the first line of (48).

Thus, we have obtained that for finite and infinite values of  $n$  it follows

$$\zeta(-2, n) = \frac{1}{6}n(n+1)(2n+1). \quad (49)$$

Analogously, by applying the same procedure again we can obtain the formulae for  $\zeta(s, n)$  for other finite integer negative points  $s$ . For instance,

$$\zeta(-3, n) = \frac{1}{4}n^2(n+1)^2, \quad (50)$$

$$\zeta(-4, n) = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1). \quad (51)$$

Note that the obtained results fit perfectly both well-known formulae for finite values of  $n$  (see [2]) and relation (20). For example, by using (20) and the obtained formulae (49), (50), and (51) with  $n = \mathbb{1}$  and  $n = \mathbb{1}/2$  we obtain (cf. (45), (46), and (47)) that

$$\begin{aligned}
\zeta(-2, \mathbb{1}) - 8\zeta(-2, \mathbb{1}/2) &= \frac{1}{6}\mathbb{1}(\mathbb{1}+1)(2\mathbb{1}+1) - \frac{1}{3}\mathbb{1}(\mathbb{1}+2)(\mathbb{1}+1) = \\
&= -0.5\mathbb{1}(\mathbb{1}+1) = \eta(-2, \mathbb{1}),
\end{aligned}$$

$$\begin{aligned}
\zeta(-3, \mathbb{1}) - 16\zeta(-3, \mathbb{1}/2) &= \frac{1}{4}\mathbb{1}^2(\mathbb{1}+1)^2 - \mathbb{1}^2(0.5\mathbb{1}+1)^2 = \\
&= -0.5\mathbb{1}^2(\mathbb{1}+3) = \eta(-3, \mathbb{1}),
\end{aligned}$$

$$\begin{aligned}
\zeta(-4, \mathbb{1}) - 32\zeta(-4, \mathbb{1}/2) &= \frac{1}{30}\mathbb{1}(\mathbb{1}+1)(2\mathbb{1}+1)(3\mathbb{1}^2 + 3\mathbb{1} - 1) - \\
\frac{1}{15}\mathbb{1}(\mathbb{1}+2)(\mathbb{1}+1)(3\mathbb{1}^2 + 6\mathbb{1} - 4) &= -0.5\mathbb{1}(\mathbb{1}+1)(\mathbb{1}^2 + \mathbb{1} - 1) = \eta(-4, \mathbb{1}).
\end{aligned}$$

## 6 Conclusion

There has been shown in this paper that, as it happens in Physics, in Mathematics the instruments used to observe mathematical objects have their own accuracy and the chosen accuracy bounds and influences results of the observation. Numeral systems used to represent numbers are among the instruments of mathematicians. When a mathematician chooses a mathematical language (an instrument), in this moment he/she chooses both a set of numbers that can be observed through the numerals available in the chosen numeral system and the accuracy of results that can be obtained. In the cases where two languages having different accuracies can be applied, it does not usually make sense to mix the languages, i.e., to compose mathematical expressions using symbols from both languages, because the result of such a mixing either has no any sense or has the lower of the two accuracies.

Two mathematical languages have been studied in the paper: (i) the traditional mathematical language using the symbol  $\infty$ ; (ii) the new language allowing one to represent different infinite and infinitesimal numbers. These languages have been applied and compared in several contexts related to the Riemann zeta function and the Dirichlet eta function. It has been discovered that the situations that can be illustrated by the following metaphor can take place.

Suppose that we have measured two distances  $A$  and  $B$  with the accuracy equal to 1 meter and we have found that both of them are equal to 25 meters. Suppose now that we want to measure them with the accuracy equal to 1 centimeter. Then, very probably, we shall obtain something like  $A = 2487$  centimeters and  $B = 2538$  centimeters, i.e.,  $A \neq B$ . Both answers,  $A = B$  and  $A \neq B$ , are correct but with different accuracies and both of them can be used successfully in different situations. For instance, if one just wants to go for a walk, then the accuracy of the answer  $A = B$  expressed in meters is sufficient. However, if one needs to connect some devices with a cable, then a higher accuracy is required and the answer expressed in centimeters should be used.

The analysis done in the paper shows that the traditional mathematical language using the symbol  $\infty$  very often does not possess a sufficiently high accuracy when one deals with problems having their interesting properties at infinity. For instance, it does not allow us to distinguish within the record  $f(x) = \sum_{i=1}^{\infty} a_i(x)$  different functions  $f_n(x) = \sum_{i=1}^n a_i(x)$  emerging for different infinite  $n$ . However, functions  $f_n(x)$  become visible if one uses the more powerful numeral systems allowing us to write down different infinite (and infinitesimal) numbers. Then, it follows that if  $a_{n+1}(x) \neq 0$  functions  $f_n(x)$  and  $f_{n+1}(x)$  are different and  $f_n(x) \neq f_{n+1}(x)$ . When  $a_{n+1}(x)$  is an infinitesimal number then the difference  $f_n(x) - f_{n+1}(x)$  is also infinitesimal, i.e., invisible if one uses the traditional mathematical language but perfectly observable through the new numeral system.

In particular, this means that it is impossible to answer to the Riemann Hypothesis because it asks a question about the behavior of something that there was supposed to be a function but, as we can see using the new numeral system distinguishing different infinite numbers, is not a function but many different functions.



Moreover, the analysis of the Riemann zeta function made traditionally does not consider various infinite and infinitesimal numbers that are crucial not only in the context of the Hypothesis but even with respect to the definition of the function itself.

Of course, there remains a possibility that the question asked in the Hypothesis has the same answer for any infinite  $n$ . Unfortunately, this is impossible due to the analysis made above and the analysis of the partial zeta functions made in [1, 3, 24].

We conclude this paper by the following remark. It is well known that the Riemann zeta function appears in many different mathematical contexts and there exist many different equivalent formulations of the Riemann hypothesis. In view of the above analysis, this can be explained again by the accuracy of the traditional mathematical language. Such manifestations indicate situations where its accuracy is not sufficient. By returning to the metaphor with the microscope, we can say that our traditional lens is too weak to distinguish different mathematical objects observed in these situations. Thus, these mathematical contexts can be viewed as very promising for obtaining new results by applying the new numeral system that does not use the symbol  $\infty$  and allows one to distinguish and to treat numerically various infinite and infinitesimal numbers.

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